Mathematical Physics III

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Syllabus

Unit III: Fourier Transform (Lectures 15)

1. Fourier Integral Theorem and Fourier Transformation
2. Fourier transform example: Trigonometric and Gaussian function
3. Representation of Dirac Delta function as a Fourier Integral
4. Fourier transformation of derivatives
5. Inverse Fourier transformation
6. Convolution Theorem (statement only)
7. Properties of Fourier Transformations: Translation, change of scale and complex conjugation.

Fourier Theorem

- Any periodic function (which follows Dirichlet conditions) having period L (*interval* $(0,L)$) can be expand in terms of linear combinations of sine and cosine terms i.e. simple harmonic motions of definite amplitude.

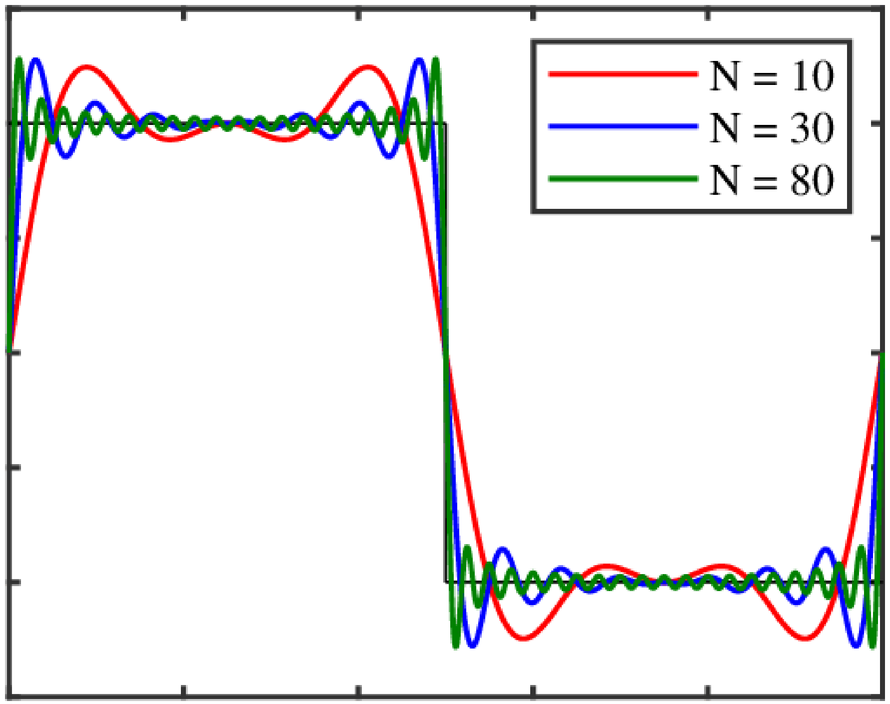
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)\right\}$$

Here, a_0, a_n and b_n are called coefficient of Fourier series.

- If you choose a time domain function $f(t)$ with interval is T (here T is the time period of a wave), then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos(\omega nt) + b_n \sin(\omega nt)\}$$



- If you choose a function $f(x)$ in $(-l, l)$ i.e. with interval is $2l$, then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2\pi nx}{2l}\right) + b_n \sin\left(\frac{2\pi nx}{2l}\right)\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{\pi nx}{l}\right) + b_n \sin\left(\frac{\pi nx}{l}\right)\right\}$$

Fourier series and integral

- Fourier series were used to present a function $f(x)$ defined of a finite interval $(-l, l)$ or $(0, l)$ etc. In this sense Fourier series is associated with periodic functions.

- Fourier integral represents a certain type of non-periodic functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$.

What is Fourier integral?

- Fourier integral is a formula for the decomposition of a non-periodic function into harmonic components whose frequencies range over a continuous set of values.

If a function f(x) satisfies the Dirichlet condition on every finite interval and if the integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

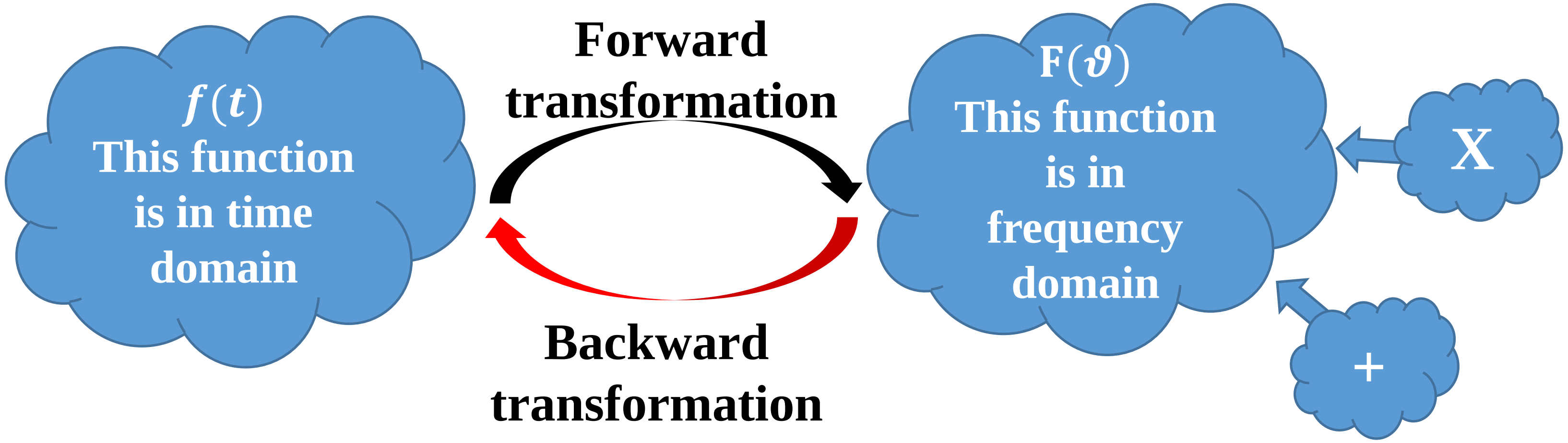
Converges, then

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \omega(t - x)\, d\omega\, dt$$

This is called **Fourier Integral Theorem**.

Fourier Transformation

- There are many classes of problems that are difficult to solve or at least quite unwieldy algebraically in their original representations.
- An integral transform "maps" an equation from its original "domain" (*as example suppose time domain*) into another domain (*as example suppose frequency domain*).
- Manipulating and solving the equation in the target domain can be easier than manipulation and solution in the original domain.
- The solution is then mapped back to the original domain with the inverse of the integral transform.



- **<u>For example:</u>**

✓ The operation of *differentiation* in the time domain corresponds to *multiplication* in the frequency domain. So some differential equations are easier to analyze in the frequency domain.

✓ Also convolution in the time domain corresponds to ordinary multiplication in the frequency domain.

Dummy variable

Shoe shop

Supervisor-1

Supervisor-2

Manager give them a task: Calculate the total production of shoe in one month

Calculate the production of shoe per hour $f(t)$

Calculate the production of shoe per day $f(x)$

After a month

Total shoe production: 1k

Total shoe production: 1k

$$\int_0^a f(t)dt = \int_0^a f(x)dx$$

Fourier Integral Theorem

- Fourier series is applicable when the $f(x)$ is a periodic function or the function obey Dirichlet conditions.
- Fourier integral is an extension of the Fourier series. When the function $f(x)$ is not periodic, then it is treated or make periodic by considering the whole x-axis (*i.e.* $f(x)$ *in* $(-\infty, \infty)$), as the period.

A. According to **Fourier integral theorem**,

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(t-x)\,d\omega\,dt$$

➢ **Proof:** We already know that the Fourier series of a function $f(x)$ in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left\{a_n \cos\left(\frac{\pi n x}{l}\right) + b_n \sin\left(\frac{\pi n x}{l}\right)\right\} \qquad \text{-----------} \ (1)$$

Here, a_0, a_n and b_n are called coefficient of Fourier series.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt\,; \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\left(\frac{n\pi t}{l}\right)dt\,; \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\left(\frac{n\pi t}{l}\right)dt$$

Fourier Integral Theorem

Substituting the value of a_0, a_n and b_n in eq. (1) we get

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\left\{\frac{1}{l}\int_{-l}^{l} f(t)\cos\left(\frac{n\pi t}{l}\right)dt\right\}\cos\left(\frac{\pi n x}{l}\right) + \sum_{n=1}^{\infty}\left\{\frac{1}{l}\int_{-l}^{l} f(t)\sin\left(\frac{n\pi t}{l}\right)dt\right\}\sin\left(\frac{\pi n x}{l}\right)$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\int_{-l}^{l}\frac{1}{l}f(t)\left\{\overset{A}{\cos\left(\frac{n\pi t}{l}\right)}\overset{B}{\cos\left(\frac{\pi n x}{l}\right)} + \overset{A}{\sin\left(\frac{n\pi t}{l}\right)}\overset{B}{\sin\left(\frac{\pi n x}{l}\right)}\right\}dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\left\{\cos\frac{n\pi}{l}(t-x)\right\}dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\left[1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{l}(t-x)\right]dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\left[cos0 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{l}(t-x)\right]dt \qquad \text{-----------} \ (2)$$

Since cosine functions are even functions i.e., $\cos(-\theta) = \cos(\theta)$, so

$$cos0 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{l}(t-x) = \sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x)$$

Fourier Integral Theorem

Now eq. (2) becomes

$$f(x)=\frac{1}{2l}\int_{-l}^{l}f(t)\left[\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x)\right]dt$$

$$=\frac{1}{2\pi}\int_{-l}^{l}f(t)\left[\frac{\pi}{l}\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x)\right]dt \quad \text{-----------} (3)$$

Let us assume that $l\to\infty$, so that we can write

$$\frac{n\pi}{l}=\omega\ (say)$$

when, $l\to\infty$, ω becomes very small

Now,

$$\Delta\omega=\omega_{n+1}-\omega_n=\frac{(n+1)\pi}{l}-\frac{n\pi}{l}=\frac{\pi}{l}$$

So when $l\to\infty$, $\Delta\omega$ becomes $d\omega$. So we can write

$$\lim_{l\to\infty}\left[\frac{\pi}{l}\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{l}(t-x)\right]=\int_{-\infty}^{\infty}\cos\omega\,(t-x)\ d\omega$$

$$=2\int_{0}^{\infty}\cos\omega\,(t-x)\ d\omega$$

Now from eq. (3)

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\left[2\int_{0}^{\infty}\cos\omega\,(t-x)\ d\omega\right]dt$$

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\omega\,(t-x)\ dt\,d\omega$$

Hence proved.

B. Fourier Sine integral

Proof: We know that,

$$\cos\omega\,(t-x)=\cos(\omega t-\omega x)$$

$$=\cos\omega t\;cos\omega x\;+\sin\omega t\;sin\omega x$$

For odd function, $\int_{-a}^{a} f(x)dx=0$

For even function, $\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$

Now, from Fourier integral theorem

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\,(\cos\omega t\;cos\omega x\;+\sin\omega t\;sin\omega x\,)\,dt\,d\omega$$

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega t\;cos\omega x\;dt\,d\omega+\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\;sin x\;dt\,d\omega \qquad \text{----------- (4)}$$

When the function $f(t)$ is odd i.e. $f(-t)=-f(t)$ the first term of equation (4) becomes zero. So we have,

$f(t)\cos\omega t$ becomes odd

$f(t)\sin\omega t$ becomes even

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\,sin\omega t\sin\omega x\,d\omega dt$$

This is called ***Fourier Sine integral***

C. Fourier cosine integral

Proof: We know that,

$$\cos\omega\,(t-x) = \cos(\omega t - \omega x)$$
$$= \cos\omega t\ cos\omega x\ + \sin\omega t\ sin\omega x$$

Now, from Fourier integral theorem

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,(\cos\omega t\ cos\omega x\ + \sin\omega t\ sin\omega x\,)\,dt\,d\omega$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega t\ cos\omega x\ dt\,d\omega + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega t\ sin x\ dt\,d\omega \qquad \text{----------- (4)}$$

Case I: When the function $f(t)$ is even i.e. $f(-t) = f(t)$ the second term of equation (4) becomes zero. So we have,

$f(t)\sin\omega t$ becomes odd

$f(t)\cos\omega t$ becomes even

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\cos\omega t\ cos\omega x\ d\omega dt$$

This is called ***Fourier Cosine integral***

Some important results

(1) $\int_0^\infty \frac{sinax}{x} = \frac{\pi}{2}$

(2) $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

(3) $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

(4) $\int_0^{2\pi} sinnxdx = 0$

(5) $\int_0^{2\pi} cosnxdx = 0$

(6) $\int_0^{2\pi} sin^2 nxdx = 0$

(7) $\int_0^{2\pi} cos^2 nxdx = 0$

(8) $\int_0^{2\pi} sinnx. sinmxdx = 0$

(9) $\int_0^{2\pi} cosnx. cosmxdx = 0$

(10) $\int_0^{2\pi} sinnx. cosmxdx = 0$

(11) $sinn\pi = 0$

(12) $cosn\pi = (-1)^n$

(13) $\int e^{Ax}\, sinBx\, dx = \frac{e^{Ax}}{A^2 + B^2}[A \sin Bx - Bcos\, Bx]$

(14) $\int e^{Ax}\, cosBx\, dx = \frac{e^{Ax}}{A^2 + B^2}[A \cos Bx + Bsin\, Bx]$

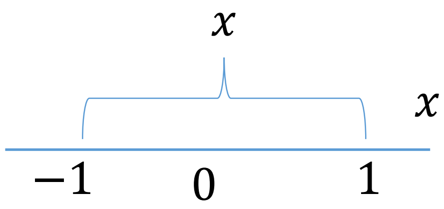
$Here\ n\ is\ an\ integer.$

Fourier Integral Problem

Q. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$\pm x < 1$ ⇒ $x < 1$, $-x < 1$ ⇒ $x < 1$, $x > -1$ ⇒



Solution:

Put, $x = -x$ in $f(x)$ we get

$$f(-x) = f(x)$$

∴ The function is an even function.

Since, we know the Fourier cosine integral of $f(x)$

$$f(x) = \frac{2}{\pi}\int_0^\infty cos\omega x \left[\int_0^\infty f(t)\cos \omega t \; dt\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty cos\omega x \left[\int_0^1 1.\cos \omega t \; dt\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty cos\omega x \left[\left|\frac{sin\omega t}{\omega}\right|_0^1\right] d\omega$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{cos\omega x \, . \, sin\omega t}{\omega} d\omega$$

$$\Rightarrow \int_0^\infty \frac{cos\omega x \, . \, sin\omega t}{\omega} d\omega = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

This is the required Fourier integral representation of the given function

Fourier Integral Problem

$$\int e^{Ax}\, sinBx\, dx = \frac{e^{Ax}}{A^2+B^2}[A\sin Bx - Bcos\, Bx]$$

Q. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} -e^{ax}, & x<0 \\ e^{-ax}, & x>0 \end{cases}$$

Solution:

Put, $x = -x$ in $f(x)$ we get

$$f(-x) = \begin{cases} -e^{-ax}, & -x<0 \\ e^{ax}, & -x>0 \end{cases} = \begin{cases} -e^{-ax}, & x>0 \\ e^{ax}, & x<0 \end{cases}$$

∴ so the function is an odd ~~even~~ function since, $f(-x) = -f(x)$.

Since, we know the Fourier sine integral of $f(x)$

$$f(x) = \frac{2}{\pi}\int_0^\infty sin\omega x\left[\int_0^\infty f(t)\sin\omega t\ dt\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty sin\omega x\left[\int_0^\infty e^{-at}.\sin\omega t\ dt\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty sin\omega x\left[\left|\frac{e^{-at}}{a^2+\omega^2}(-asin\omega t - \omega sin\omega t)\right|_0^\infty\right]d\omega$$

$$f(x) = \frac{2}{\pi}\int_0^\infty sin\omega x\left[\frac{\omega}{a^2+\omega^2}\right]d\omega$$

$$\Rightarrow \int_0^\infty sin\omega x\,(\frac{\omega}{a^2+\omega^2})d\omega = \frac{\pi}{2}f(x)$$

$$= \begin{cases} \frac{\pi}{2}(-e^{ax}), x<0 \\ \frac{\pi}{2}(-e^{ax}), x<0 \end{cases}$$

This is the required Fourier integral representation of the given function.

Fourier

Fourier series → Periodic function is expressed as linear combinations of sine and cosine terms.

Fourier Integral → It is an extension of the Fourier series having period $(-\infty, \infty)$.

Fourier Transformation → It is an integral transform which “maps” an equation from its original “domain” into another domain for the ease of handling.

Fourier Complex Integral

- **Proof:** We already know that

For odd function, $\int_{-a}^{a} f(x)dx = 0$

$$\therefore \qquad \int_{-\infty}^{\infty} \sin\omega\,(t-x)d\omega = 0$$

Since, $\sin\omega\,(t-x)$ is a odd function

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \sin\omega\,(t-x)d\omega = 0$$

$$\Rightarrow \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \sin\omega\,(t-x)d\omega = 0$$

Multiplying by i

Fourier integral theorem

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega\,(t-x)\ dt\,d\omega$$

Now adding this expression with the Fourier integral theorem, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega\,(t-x)\,dt\,d\omega + \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \sin\omega\,(t-x)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} [\cos\omega\,(t-x) + i\sin\omega\,(t-x)]\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)dt \int_{-\infty}^{\infty} e^{i\omega(t-x)}\,d\omega \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\omega(t-x)}\,d\omega\,dt$$

This is called Fourier complex integral

Fourier Transformation

- We have found the expression for Fourier complex integral as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\omega(t-x)}\, d\omega\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} d\omega \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} . F(\omega) d\omega$$

Where,

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

Here, $F(\omega)$ **is called the Fourier transformation of** $f(x)$.

And, $f(x)$ **is called the inverse Fourier transformation of** $F(\omega)$.

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega t}\, d\omega$$

$e^{i\omega t} e^{-i\omega t}$=1

$$\frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$$

Fourier sine transformation:

We know the Fourier sine integral

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\, sin\omega t \sin \omega x\, d\omega dt$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \sin \omega x\, d\omega \left[\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\, sin\omega t\, dt\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \sin \omega x\, d\omega\, F_s(\omega)$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s(\omega)\, \sin \omega x\, d\omega$$

Where, $\boldsymbol{F_s(\omega)} = \sqrt{\frac{2}{\pi}}\int_0^\infty \boldsymbol{f(t)\, sin\omega t\, dt}$ is called the Fourier sine transformation of $f(x)$.

Fourier Transformation

Fourier cosine transformation:

We know the Fourier cosine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\, cos\omega t \cos\omega x \, d\omega dt$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos\omega x \, d\omega \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\, cos\omega t \, dt\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty cos\omega x \, d\omega \, F_c(\omega)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\omega) \cos\omega x \, d\omega$$

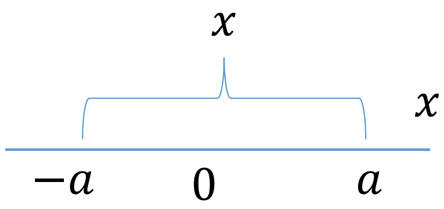
Where, $\boldsymbol{F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\, cos\omega t \, dt}$ is called the Fourier cosine transformation of $f(x)$.

Fourier Transformation

Q. Find the Fourier transformation of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$\pm x < a$ ⇒ $x < a$, $-x < a$ ⇒ $x < a$, $x > -a$ ⇒



Solution:

Put, $x = -x$ in $f(x)$ we get

$$f(-x) = f(x)$$

∴ The function is an even function.

Since, we know the Fourier cosine transformation of $f(x)$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\, cos\omega t\, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a 1. cos\omega t\, dt$$

$$= \sqrt{\frac{2}{\pi}} \left| \frac{sin\omega t}{\omega} \right|_0^a$$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \left(\frac{sin\omega a}{\omega}\right)$$

This is the required Fourier transformation of $f(x)$.

Fourier Transformation

Q. Find the Fourier transformation of $f(t) = e^{-|t|}$. Using this, prove that, $\int_0^\infty \frac{cos\omega t}{(1+\omega^2)} d\omega = \frac{\pi}{2} e^{-|t|}$

Solution:

Put, t= $-t$ in $f(t)$ we get

$$f(-t) = f(t)$$

∴ The function is an even function.

Since, we know the Fourier cosine transformation of $f(x)$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\, cos\omega t\, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t}.\, cos\omega t\, dt$$

$$= real\ part\ of \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t}.\, e^{i\omega t} dt$$

$$= real\ part\ of \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t(1-i\omega)}\, dt$$

$$F_c(\omega) = real\ part\ of \sqrt{\frac{2}{\pi}} \frac{1}{(1-i\omega)}$$

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$= real\ part\ of \sqrt{\frac{2}{\pi}} \frac{(1+i\omega)}{(1+\omega^2)}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{(1+\omega^2)}$$

This is the required Fourier transformation of $f(t)$.

We know the Fourier cosine integral

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\omega)\ \cos\omega t\, d\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{(1+\omega^2)} \cos\omega t\, d\omega$$

Fourier Transformation

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{(1+\omega^2)} \cos \omega t \, d\omega$$

$$\Rightarrow \qquad e^{-|t|} = \frac{2}{\pi} \int_0^\infty \frac{\cos \omega t}{(1+\omega^2)} d\omega$$

$$\Rightarrow \int_0^\infty \frac{cos\omega t}{(1+\omega^2)} d\omega = \frac{\pi}{2} e^{-|t|}$$

Hence proved.

1. <u>Linear property</u>:

If $\boldsymbol{F_1(s)}$ *(or* $F_1(\omega)$*)* and $\boldsymbol{F_2(s)}$ *(or* $F_2(\omega)$*)* are Fourier transformation of $\boldsymbol{f_1(x)}$ and $\boldsymbol{f_2(x)}$ respectively, then

$$\boldsymbol{F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)}$$ Where, a and b are constant.

<u>Proof</u>: We know that

$$F_1(f_1(x)) = F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{isx}\, dx$$

$\omega \Leftrightarrow s$

t$\Leftrightarrow x$

And $$F_2(f_2(x)) = F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) e^{isx}\, dx$$

Now,

$$F[af_1(x) + bf_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af_1(x) + bf_2(x)]\ e^{isx}\, dx$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)\ e^{isx}\, dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)\ e^{isx}\, dx$$

$$= aF_1(s) + bF_2(s)$$

Hence proved.

2. <u>Change of scale property</u>:

If $\boldsymbol{F(s)}$ is the complex Fourier transformation of $\boldsymbol{f(x)}$, then $\boldsymbol{F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$ Where, a is constant.

<u>Proof</u>: We know that

$$F\big(f(x)\big) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

Now,

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\ e^{isx}\, dx$$

Put $ax = t$

$\Rightarrow \mathrm{dx} = \frac{dt}{a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{is\frac{t}{a}} \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{it(\frac{s}{a})}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

Hence proved.

3. <u>Shifting property</u>:

If $\boldsymbol{F}(\boldsymbol{s})$ is the complex Fourier transformation of $\boldsymbol{f}(\boldsymbol{x})$, then $\boldsymbol{F}[\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{a})] = \boldsymbol{e}^{\boldsymbol{isa}}\boldsymbol{F}(\boldsymbol{s})$ Where, a is constant.

<u>Proof</u>: We know that

$$F\big(f(x)\big) = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

Now,

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\ e^{isx}\,dx$$

Put $x - a = t$

$\Rightarrow$ $\mathrm{dx} = dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ e^{is(a+t)}\,dt$$

$$= \frac{e^{isa}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ e^{ist}\,dt$$

$$= e^{isa}F(s)$$

Hence proved.

4. <u>Exponential multiplication</u>:

$$\boldsymbol{F}\left[\boldsymbol{e}^{\boldsymbol{iax}}\boldsymbol{f}(\boldsymbol{x})\right] = \boldsymbol{F}(\boldsymbol{s}+\boldsymbol{a})$$

<u>Proof</u>: We know that

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)e^{isx}\,dx$$

Now,

$$F\left[e^{iax}f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)\ e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x}f(x)\,dx$$

$$= F(s+a)$$

5. <u>Complex conjugation property</u>:

If $\boldsymbol{F}(\boldsymbol{s})$ is the complex Fourier transformation of $\boldsymbol{f}(\boldsymbol{x})$, then $\boldsymbol{F}[\boldsymbol{f}^*(\boldsymbol{x})] = \boldsymbol{F}^*(-\boldsymbol{s})$

<u>Proof</u>: We know that

$$F\big(f(x)\big) = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

Taking complex conjugate on both sides, we have

$$F^*(s) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx\right]^*$$

$$\Rightarrow F^*(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(x)e^{-isx}\,dx$$

$$\Rightarrow F^*(-s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(x)e^{isx}\,dx$$

$$= F[f^*(x)]$$

6. <u>Derivative of Fourier transformation</u>:

If $\boldsymbol{F(s)}$ is the complex Fourier transformation of $\boldsymbol{f(x)}$ i.e. $F[f(x)] = F(s)$, then $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$

<u>Proof</u>: We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

Differentiating w.r.t. s both side, n-times, we get

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n e^{isx} f(x)\, dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x)\, e^{isx} dx$$

$$= (i)^n F[x^n f(x)]$$

$$or\ F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$$

4. Fourier transformation of derivatives

If $F(s)$ is the complex Fourier transformation of $f(x)$ i.e. then the Fourier transform, $F_1(s)$ of the first derivative $f^{/}(x)$ may be written as

$$F_1(s) = F\left[f^{/}(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{/}(x)\, e^{isx}\, dx$$

V U

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{df}{dx}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx} f(x)\right]_{-\infty}^{\infty} - is\int_{-\infty}^{\infty} f(x) e^{isx}\, dx\right\}$$

Where $f(x)$ is a well behaved function such that it reduces to zero at a much faster rate than e^{isx} tends to zero as x→ ∞. For such function the above equation reduce to

$$F_1(s) = -is\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= -isF(s)$$

Similarly we have

$$F_2(s) = F\left[f^{//}(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^2 f}{dx^2}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx} f^{/}(x)\right]_{-\infty}^{\infty} - is\int_{-\infty}^{\infty} f^{/}(x) e^{isx}\, dx\right\}$$

$$= -is\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{/}(x) e^{isx}\, dx$$

$$= (-is)[-isF(s)]$$

$$= (-is)(-is)F(s)$$

$$= (-is)^2\, F(s)$$

Similarly we have

$$F_n(s) = (-is)^n\, F(s)$$

4. Fourier transformation of derivatives

Q. Prove the following

$$1.\ F_c[f^{/}(x)] = sF_s(s) - \sqrt{\frac{2}{\pi}}f(0)$$

$$2.\ F_c[f^{//}(x)] = -s^2F_c(s) - \sqrt{\frac{2}{\pi}}f^{/}(0)$$

$$3.\ F_s[f^{/}(x)] = -sF_c(s)$$

$$4.\ F_s[f^{//}(x)] = -s^2F_s(s) + \sqrt{\frac{2}{\pi}}s\,f(0)$$

45.13 FOURIER TRANSFORM OF DERIVATIVES

We have already seen that,

$$F\{f^n(x)\} = (-is)^n\,F(s)$$

(i) $\therefore\ F\left(\frac{\partial^2 u}{\partial x^2}\right) = (-is)^2 F\{u(x)\} = -s^2\bar{u}$ where $\bar{u}$ is Fourier transform of u w.r.t. x.

(ii) $$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}f(o) + s\,F_s(s)$$

$$\text{L.H.S.} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cdot\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,d\{f(x)\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\{f(x)\cos sx\}_0^\infty + s\int_0^\infty f(x)\sin sx\,dx\right]$$

$$= s\,F_s(s) - \sqrt{\frac{2}{\pi}}f(0) \quad \text{assuming } f(x)\to 0 \text{ as } x\to\infty$$

(iii) $$F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\,d\,[f(x)] = \sqrt{\frac{2}{\pi}}\left[\{f(x)\sin sx\}_0^\infty - s\int_0^\infty f(x)\cos sx\,dx\right]$$

$$= -s\,F_c(s)$$

(iv) $$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,d\,[f'(x)] = \sqrt{\frac{2}{\pi}}\left[\{f'(x)\cos sx\}_0^\infty + s\int_0^\infty f'(x)\sin sx\,dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + s\,F_s\{f'(x)\} = -s^2\,F_c(s) - \sqrt{\frac{2}{\pi}}f'(0) \quad \text{assuming } f(x),\ f'(x)\to 0 \text{ as } x\to\infty$$

(v) $$F_s\{f''\}(x) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty \sin sx\,d[f'(x)]\right] = \sqrt{\frac{2}{\pi}}\left[(f'(x)\sin sx)_0^\infty - s\int_0^\infty f'(x)\cos sx\,dx\right]$$

$$= -s\,F_c\{f'(x)\} = -s\left[s\,F_s(s) - \sqrt{\frac{2}{\pi}}f(0)\right]$$

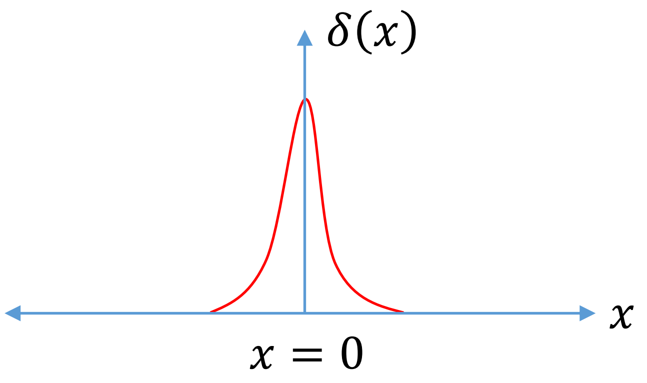
$$= -s^2\,F_s(s) + \sqrt{\frac{2}{\pi}}s\,f(0) \quad \text{assuming } f(x),\ f'(x)\to 0 \text{ as } x\to\infty.$$

- In mathematics, the **Dirac delta function** (**δ function**), also known as the **unit impulse** symbol is a generalized function or distribution over the real numbers, whose value is zero everywhere except at zero, and whose integral over the entire real line is equal to one

1. $\delta(x) = 0$; when $x \neq 0$

   $\neq 0$; when $x = 0$

2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$; the area under the curve becomes 1.



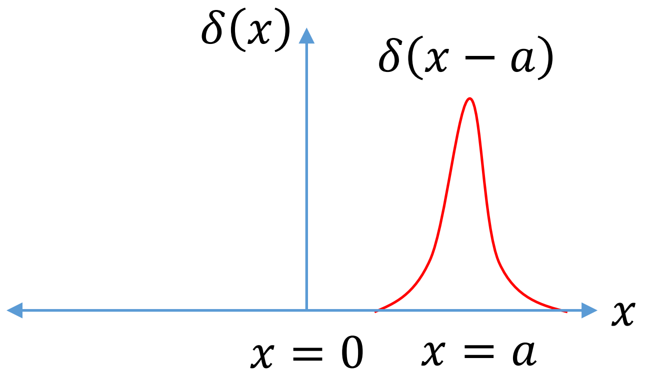
<u>Properties:</u>

1. $\delta(-x) = \delta(x)$; it is a even function

2. $x\, \delta(x) = 0$; since when $x \neq 0, \delta(x) = 0$

   when $x = 0, \delta(x) \neq 0$

3. $f(x)\, \delta(x) = f(0)\, \delta(x)$ ⟹ 3. $f(x)\, \delta(x-a) = f(a)\, \delta(x-a)$



4. $\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = \int_{-\infty}^{\infty} f(0)\, \delta(x)\, dx$

$$= f(0) \int_{-\infty}^{\infty} \delta(x)\, dx = f(0)$$ ⟹ 4. $\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$

3. Fourier transformation of Dirac delta function

If $\boldsymbol{F(s)}$ is the complex Fourier transformation of $\boldsymbol{f(x)}$, them we can write

$\delta(x)$

$$F(s) = F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\Rightarrow \quad F\left(\delta(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)\, e^{isx}\, dx$$

$$\Rightarrow \quad F\left(\delta(x)\right) = \frac{1}{\sqrt{2\pi}} e^{0} \quad = \frac{1}{\sqrt{2\pi}}$$

Now, the inverse Fourier transformation

$$\delta(x) = F^{-1}\left[F\left(\delta(x)\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-isx}\, ds$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds$$

Similarly, $\delta(x-a)$

$$F\left(\delta(x-a)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)\, e^{isx}\, dx$$

$$\Rightarrow \quad F\left(\delta(x-a)\right) = \frac{1}{\sqrt{2\pi}} e^{isa}$$

Now, the inverse Fourier transformation

$$\delta(x-a) = F^{-1}\left[F\left(\delta(x-a)\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isa} e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(x-a)}\, ds$$

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(x-a)}\, ds$$

In general

$$\int_{-\infty}^{\infty} e^{-is(x-a)}\, ds = 2\pi\, \delta(x-a)$$

3. Representation of Dirac Delta function as a Fourier Integral

- We have found the expression for Fourier complex integral as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\omega(t-x)}\, d\omega\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} e^{i\omega(t-x)}\, d\omega\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(x-t)}\, d\omega\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\delta(x-t)\, dt$$

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is(x-a)}\, ds$$

$$f(x) = \int_{-\infty}^{\infty} \delta(x-t)\, f(t) dt$$

This is the required Fourier integral of Dirac Delta function.

- Fourier transform of $\boldsymbol{\sin \omega_0 t}$:

We know that

$$\sin \omega_0 t = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$

Therefore,

$$F(\sin \omega_0 t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin \omega_0 t \,.\, e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right) .\, e^{ist}\, dt$$

$$= \frac{1}{2i\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{i\omega_0 t} .\, e^{ist}\, dt - \int_{-\infty}^{\infty} e^{-i\omega_0 t} .\, e^{ist}\, dt \right]$$

$$= \frac{1}{2i\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-it(-\omega_0 - s)} dt - \int_{-\infty}^{\infty} e^{-it(\omega_0 - s)} dt \right]$$

$$= \frac{1}{2i\sqrt{2\pi}} [2\pi\, \delta(-\omega_0 - s) - 2\pi\, \delta(\omega_0 - s)]$$

$$\int_{-\infty}^{\infty} e^{-is(x-a)}\, ds = 2\pi\, \delta(x-a)$$

$$= \frac{1}{i}\sqrt{\frac{\pi}{2}}\, [\delta(-\omega_0 - s) - \delta(\omega_0 - s)] \quad = \frac{1}{i}\sqrt{\frac{\pi}{2}}\, [\delta(\omega_0 + s) - \delta(\omega_0 - s)]$$

$$\delta(-x) = \delta(x)$$

- Fourier transform of **$\cos \omega_0 t$** :

We know that

$$\cos \omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$

Therefore,

$$F(\cos \omega_0 t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos \omega_0 t \,.\, e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right) .\, e^{ist}\, dt$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{i\omega_0 t} . e^{ist}\, dt + \int_{-\infty}^{\infty} e^{-i\omega_0 t} . e^{ist}\, dt \right]$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-it(-\omega_0 - s)} dt + \int_{-\infty}^{\infty} e^{-it(\omega_0 - s)} dt \right]$$

$$= \frac{1}{2\sqrt{2\pi}} [2\pi\, \delta(-\omega_0 - s) + 2\pi\, \delta(\omega_0 - s)]$$

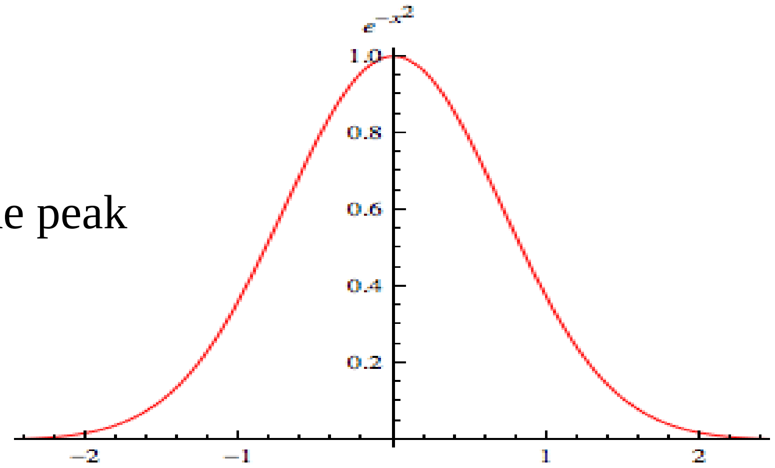
$$\int_{-\infty}^{\infty} e^{-is(x-a)}\, ds = 2\pi\, \delta(x-a)$$

$$= \sqrt{\frac{\pi}{2}}\, [\delta(-\omega_0 - s) + \delta(\omega_0 - s)] \qquad = \sqrt{\frac{\pi}{2}}\, [\delta(\omega_0 + s) + \delta(\omega_0 - s)]$$

$$\delta(-x) = \delta(x)$$

2. Fourier transform of Gaussian function

- The normalized form of Gaussian distribution function



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}}$$

$\frac{1}{\sigma\sqrt{2\pi}} \rightarrow$ Height of the curve peak

$b \rightarrow$ The position of the center of the peak

$\sigma \rightarrow$ Standard deviation

When b=0

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$-\infty < x < \infty$$

Therefore,

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} . e^{isx} \; dx$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{\left(-\frac{x^2}{2\sigma^2} + isx\right)} \; dx$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left[\left(-\frac{1}{2\sigma^2}\right) \times (x^2 - 2\sigma^2 isx)\right] dx$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} \exp\left[\left(-\frac{1}{2\sigma^2}\right) \times (x^2 - 2\sigma^2 isx + (\sigma^2 is)^2 - (\sigma^2 is)^2)\right] dx$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2\sigma^2}\right)(-\sigma^2 is)^2} e^{\left(-\frac{1}{2\sigma^2}\right)\left(x^2 - 2\sigma^2 isx + (\sigma^2 is)^2\right)} \; dx$$

$$= \frac{e^{\left(-\frac{\sigma^2 s^2}{2}\right)}}{2\pi\sigma} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2\sigma^2}\right)(x - \sigma^2 is)^2} \; dx$$

Applying the transformation
$x - \sigma^2 is \rightarrow u$
$du = dx$

$$= \frac{e^{\left(-\frac{\sigma^2 s^2}{2}\right)}}{2\pi\sigma} \int_{-\infty}^{\infty} e^{\left(-\frac{u^2}{2\sigma^2}\right)} \; du$$

$$= e^{\left(-\frac{\sigma^2 s^2}{2}\right)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{u^2}{2\sigma^2}\right)} \; du$$

=1, since the Gaussian function is normalised

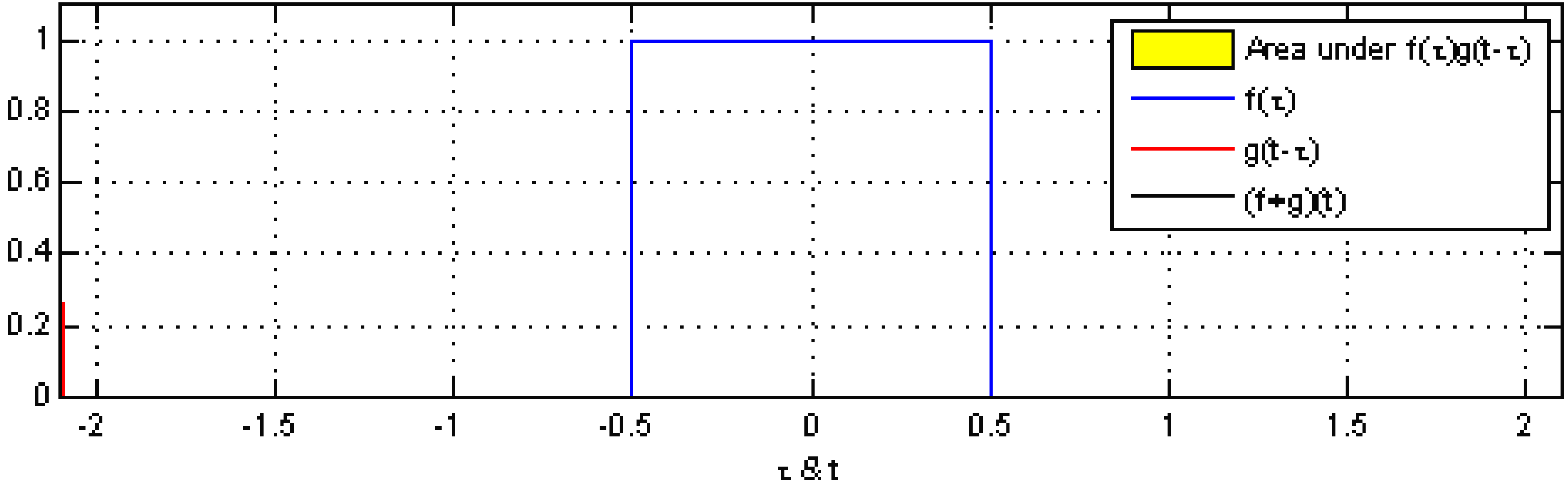
$$= \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{\sigma^2 s^2}{2}\right)}$$

Definition:

The convolution of $\boldsymbol{f}$ and $\boldsymbol{g}$ is written as $\boldsymbol{f} * \boldsymbol{g}$, denoting the operator with the symbol $*$. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)$$

While the symbol $\boldsymbol{t}$ is used above, it need not represent the time domain. But in that context, the convolution formula can be described as the area under the function $f(\tau)$ weighted by the function $g(-\tau)$ shifted by amount t. As t changes, the weighting function $g(t - \tau)$ emphasizes different parts of the input function $f(\tau)$.



6. Convolution Theorem (statement only)