Mathematical Physics III

PHY-HC-4016

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Syllabus

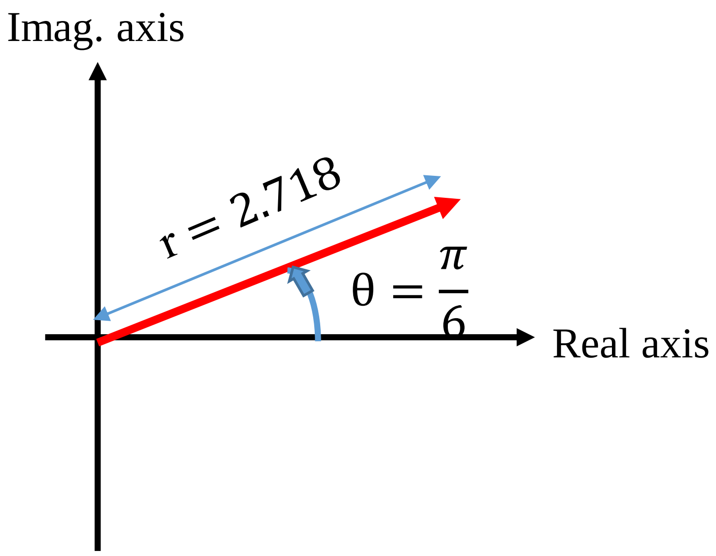
Unit I: Complex Analysis (Lectures 10)

1. Function of complex variables
2. Analytic and Cauchy-Riemann conditions
3. Example of analytic functions
4. Singular functions: Poles and branch points
5. Order of singularity

Q. Plot the number $e^{(1+i\frac{\pi}{6})}$. (2017, mark: 1)

$$= e^{(1+i\frac{\pi}{6})}$$
$$= e^{1} \times e^{(i\frac{\pi}{6})}$$
$$= e^{1} \times (cos\frac{\pi}{6} + isin\frac{\pi}{6})$$

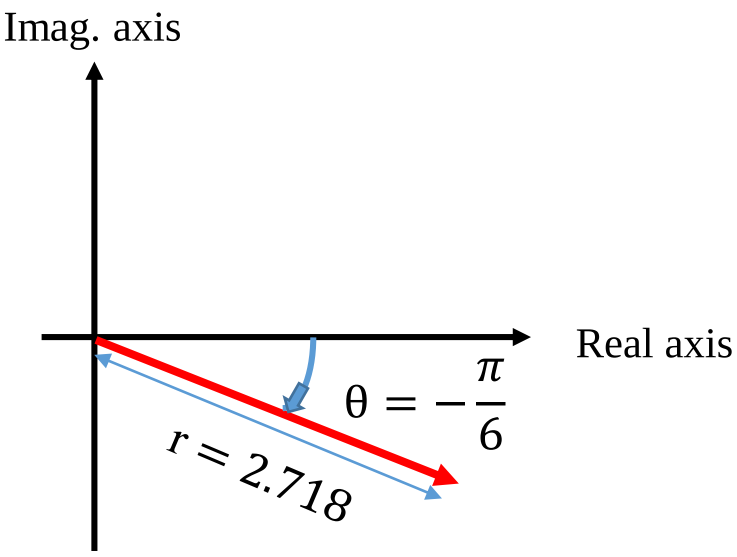
Here, r = e^{1}= 2.718 and argument, $\theta = \frac{\pi}{6}$



Q. Plot the number $e^{(1-i\frac{\pi}{6})}$. (2015, mark: 1)

$$= e^{(1-i\frac{\pi}{6})}$$
$$= e^{1} \times e^{(-i\frac{\pi}{6})}$$
$$= e^{1} \times (cos\frac{\pi}{6} - isin\frac{\pi}{6})$$

Here, r = e^{1}= 2.718 and argument, $\theta = -\frac{\pi}{6}$

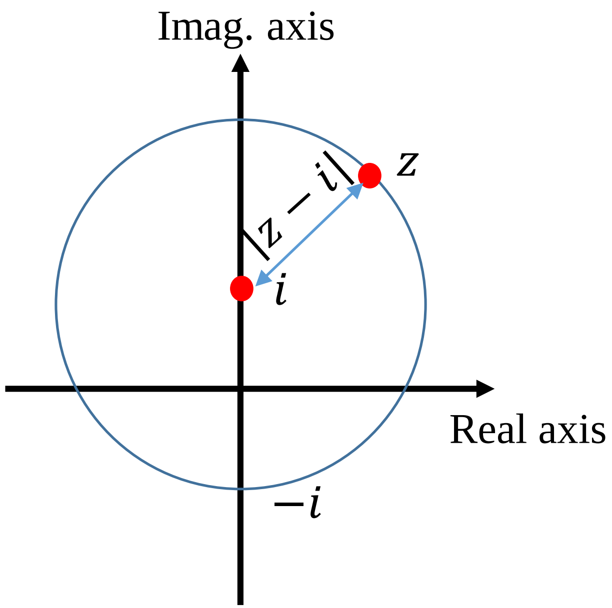


Q. What does the equation $|z - i| = 2$ represent? (2015, mark: 1)

Ans.: Here Z is any complex number.

$$|z - (i)| = 2$$

Complex number



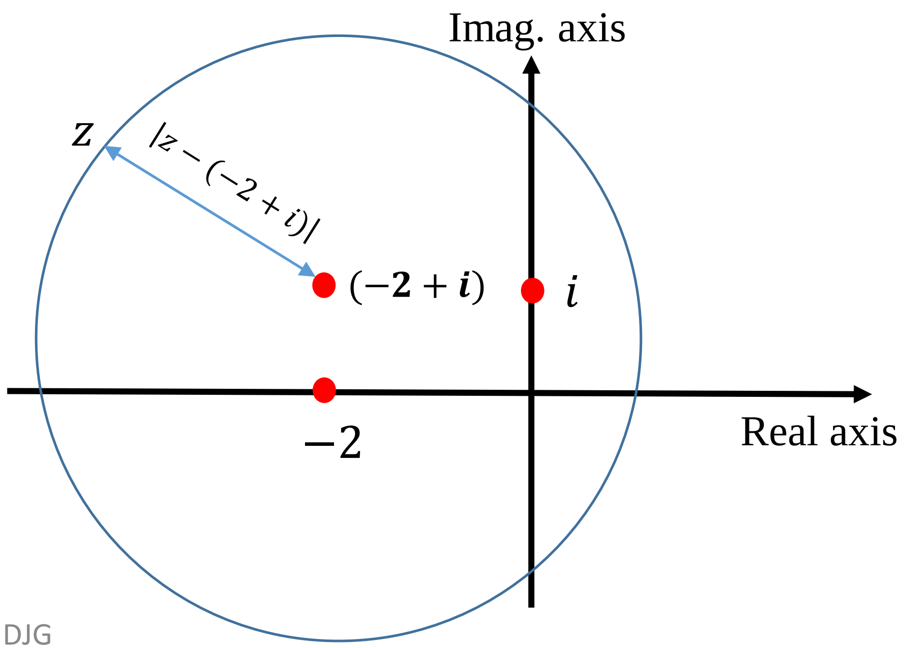
So, the given eq. represent a circle of radius ***2*** at center ***i*.**

Q. Find an equation for (a) a circle of radius 4 with center at (-2, 1) or $(-2 + i)$. (Page no. 15, Book: Schaum's outline)

Ans.:The center can be represented by the complex no. (-2 + i). If Z is any point on the circle, the distance from Z to $2 + i$ is

$$|z - (-2 + i)| = 4$$

This is the required equation.



- A symbol, such as Z, which can stand for any one of a set of complex numbers is called a complex variable.

$$Z = x + iy$$

Q. What do you mean by a function?

Ans.: A function is a **procedure** which gives a unique output for any suitable input.
The set of suitable input is called as **domain** of the function while the set of outputs which are possible is called the **range** of the function.

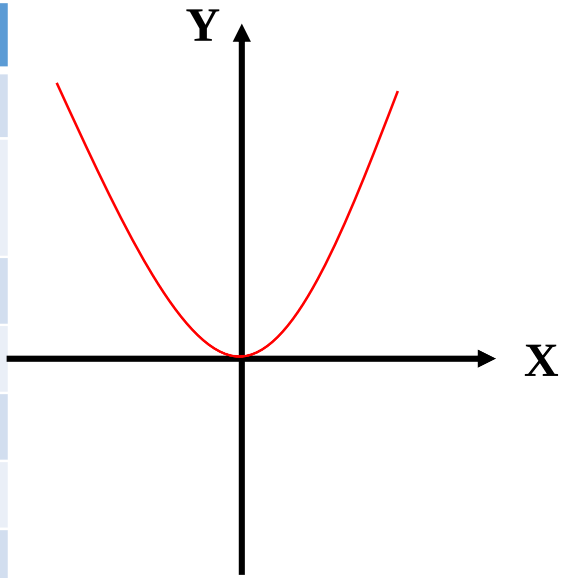
Real function: Let, y is a function of x and this is written as

$$y = f(x)$$

For example, $y = x^2$

Mapping

x	$y=x^2$
Domain	Range
Independent variable	Dependent variable
3	9
2	4
0	0
-2	4
3	9



$Z = x + iy$ Or $Z = (x, y)$ → *Function machine* → $W = f(Z) = u + iv$ Or $W = (u, v)$

- Now, for example, let us consider a complex variable

$$W = f(Z) = Z^2$$

$$=> u + iv = (x + iy)^2$$

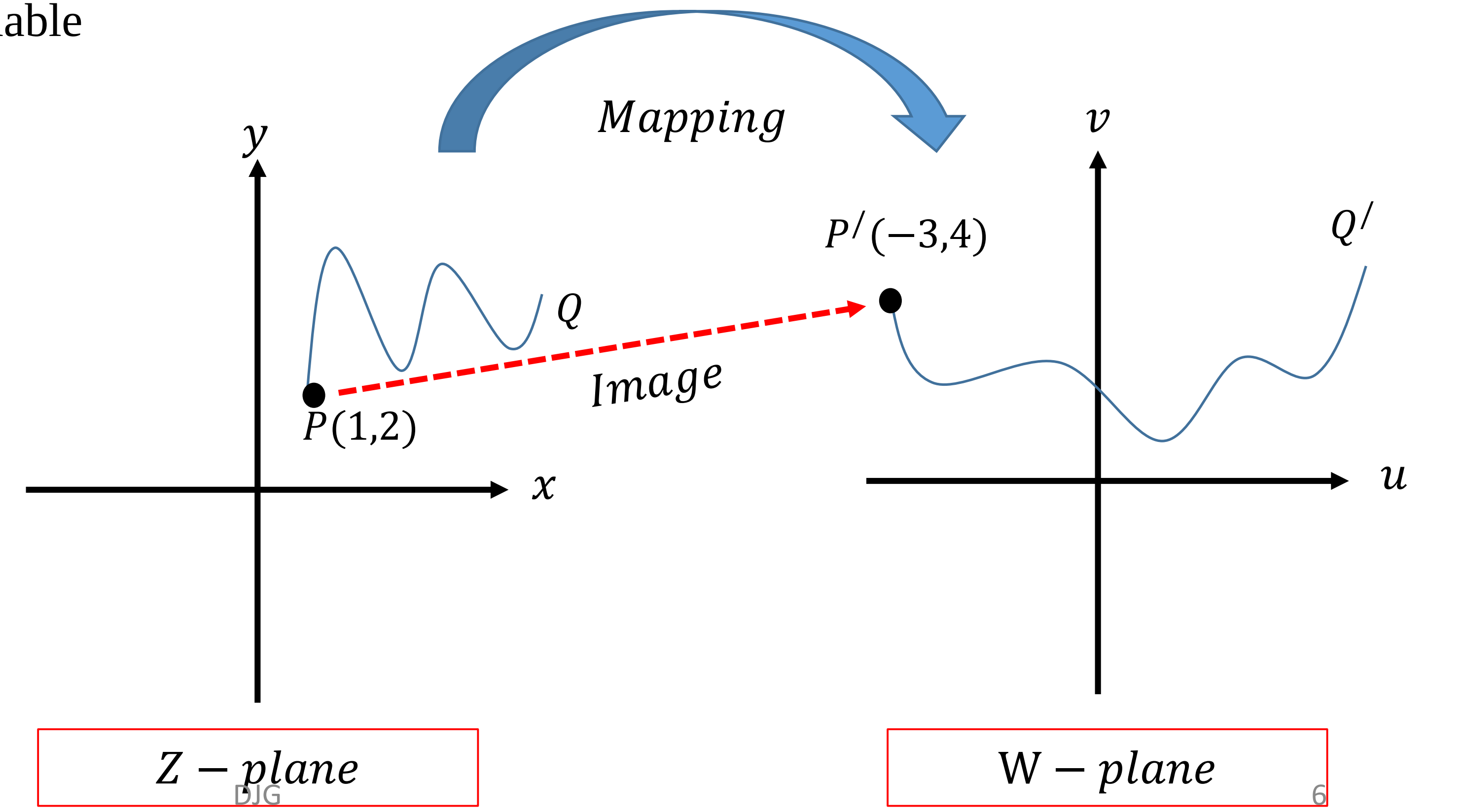
$$=> u + iv = (x^2 - y^2) + i(2xy)$$

Therefore, we get

$$u = x^2 - y^2$$

$$v = 2xy$$

So, u *and* v *is a function of* (x, y)
$i.e.\ u(x, y)$ *and* $v(x, y)$



Q. Write the polar form of the given complex number.

$$Z = (-1 + i)$$

$$z = r(cos\theta + isin\theta)$$

Q. Find the roots and locate them graphically. (*pg. no. 23*)

$$(-1+i)^{1/3}$$

Solution: We can write

$$W = z^{1/3}$$

Where, Z is a complex no. i.e., $\mathbf{Z = -1 + i}$ and **W is a function of Z.**

Since, the polar form of, $Z = -1 + i$

$$Z = -1 + i = \sqrt{2}\{cos(3\pi/4 + 2k\pi) + i\ sin(3\pi/4 + 2k\pi)\}$$

$$Z^{1/3} = (-1+i)^{1/3} = 2^{1/6}\left\{cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\ sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right\}$$

$$If\ k = 0, \boldsymbol{Z_1 = 2^{1/6}(cos\,\pi/4 + i\,sin\,\pi/4)}$$

$$If\ k = 1, \boldsymbol{Z_2 = 2^{1/6}(cos\,11\pi/12 + i\,sin\,11\pi/12)}$$

$$If\ k = 2, \boldsymbol{Z_3 = 2^{1/6}(cos\,19\pi/12 + i\,sin\,19\pi/12)}$$

So these are the required roots.

The root of a number x is another number, which when multiplied by itself a given number of times, equals x.

$$y = \sqrt{x}$$

Suppose x=4, Now

$$y = \sqrt{4}$$

∴ So roots of 4 is ± 2

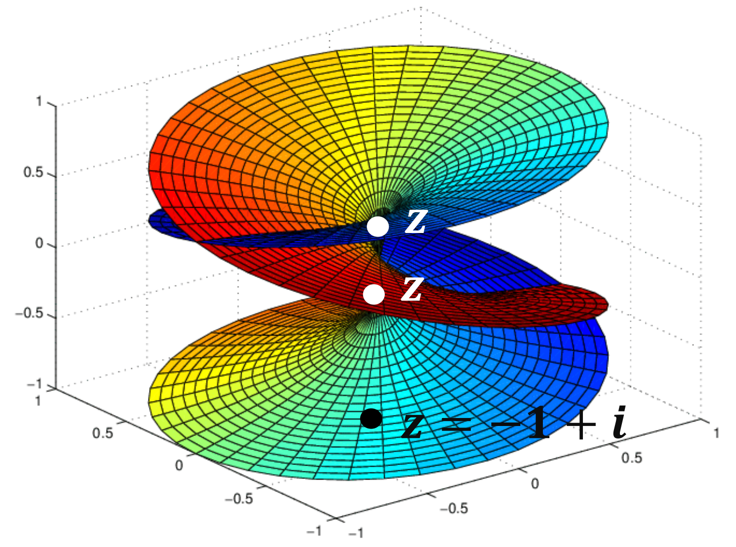
Roots of complex no.

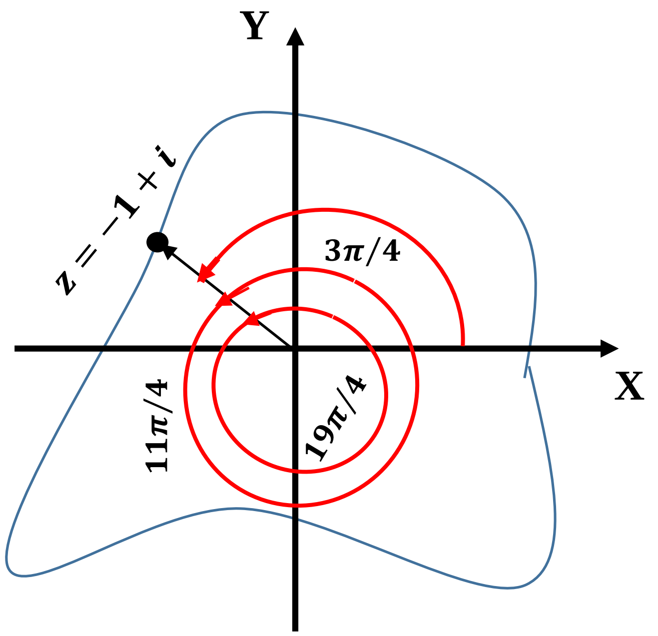
$$W = Z^{1/3} = (-1+i)^{1/3} = 2^{1/6}\left\{\cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right\}$$

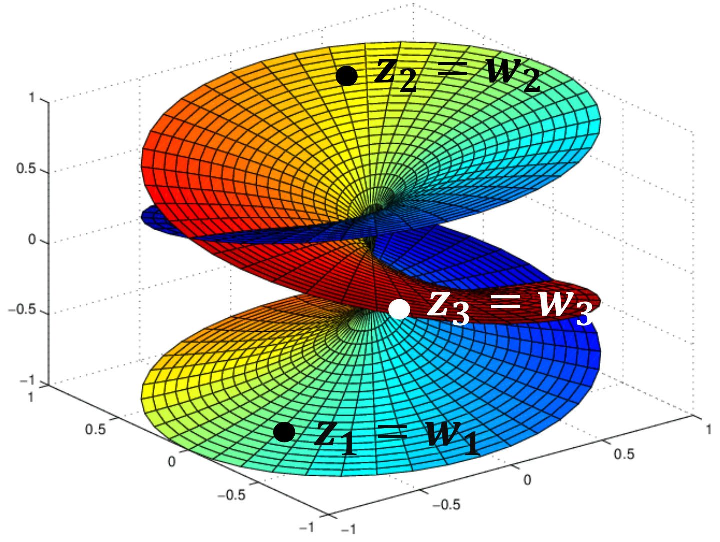
$$If\ k = 0, Z_1 = 2^{1/6}(\cos\pi/4 + i\sin\pi/4)$$

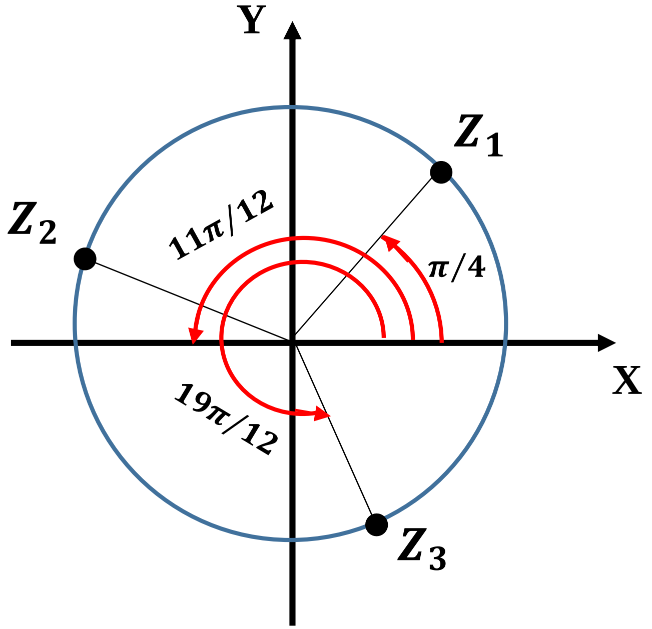
$$If\ k = 1, Z_2 = 2^{1/6}(\cos 11\pi/12 + i\sin 11\pi/12)$$

$$If\ k = 2, Z_3 = 2^{1/6}(\cos 19\pi/12 + i\sin 19\pi/12)$$





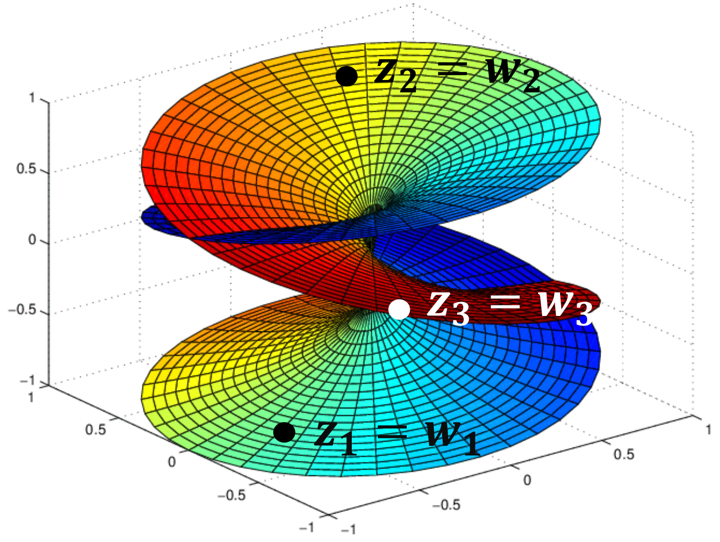


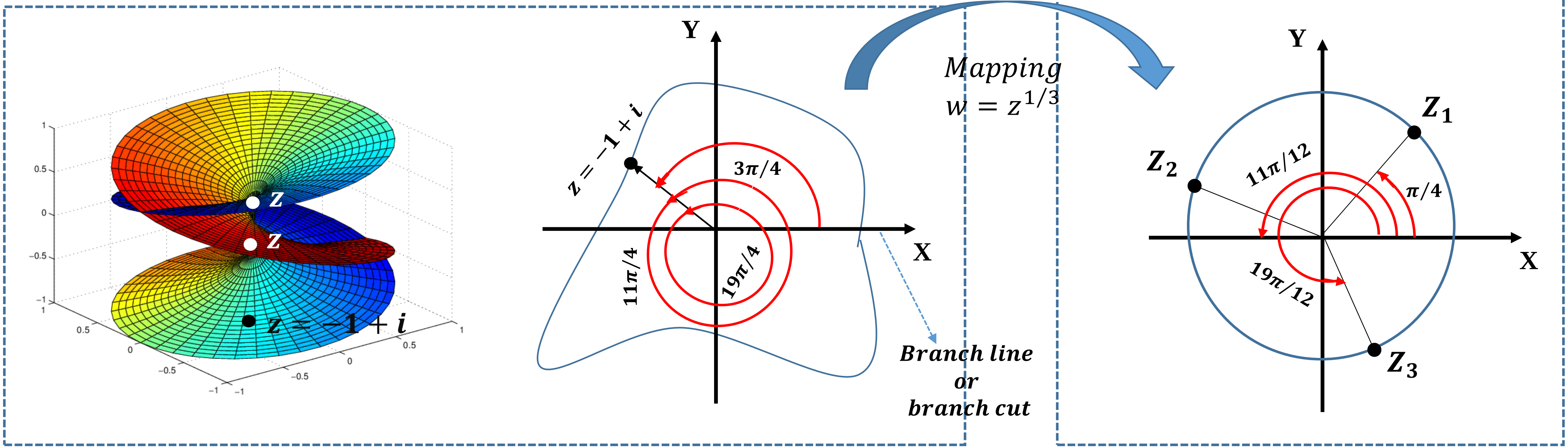


- Roots of complex number (pg. no 23)
- Riemann surface (pg. no 46)
- Function of complex variable
- Branch line (pg. no. 45)
- Branch point (pg. no 45)

Roots of complex no.

- Each sheet corresponds to a branch of the function and on each sheet the function is single-valued.
- The concept of Riemann surfaces has the advantage that the various values of multiple-valued functions are obtained in a continuous fashion.
- For example, for the function $z^{1/3}$ the Riemann surface has 3 sheets; for ln z, the Riemann surface has infinitely many sheets.



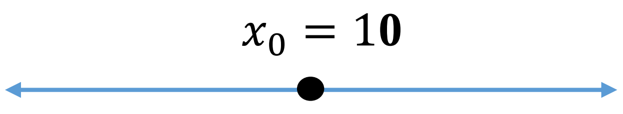


$Z - plane$

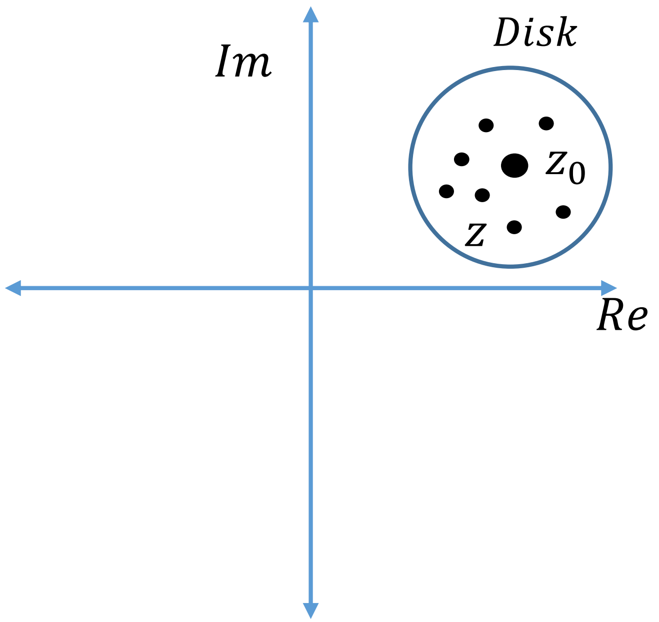
$W - plane$

Neighborhoods: A delta, or δ, neighborhood of a point z_0 is the set of all points z such that $|z - z_0| < \delta$, where δ is any given positive number.

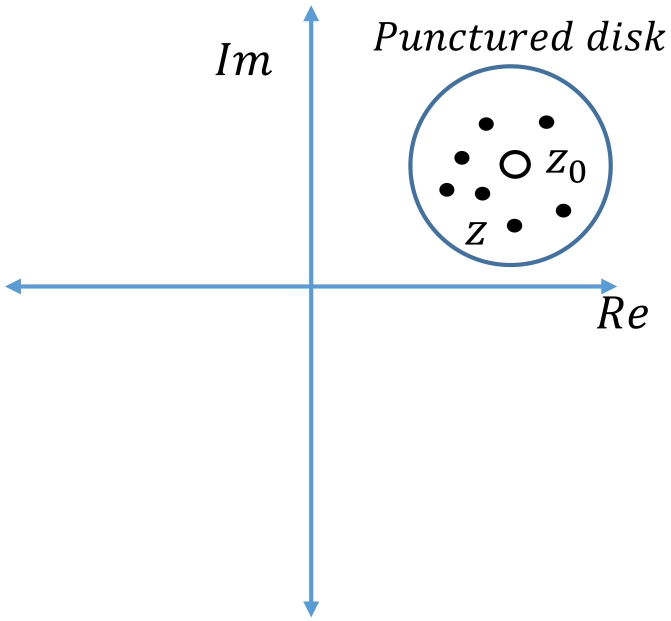
A deleted δ neighborhood of z_0 is a neighborhood of z_0 in which the point z_0 is omitted, i.e., $0 < |z - z_0| < \delta$.



Real number line



Z-plane



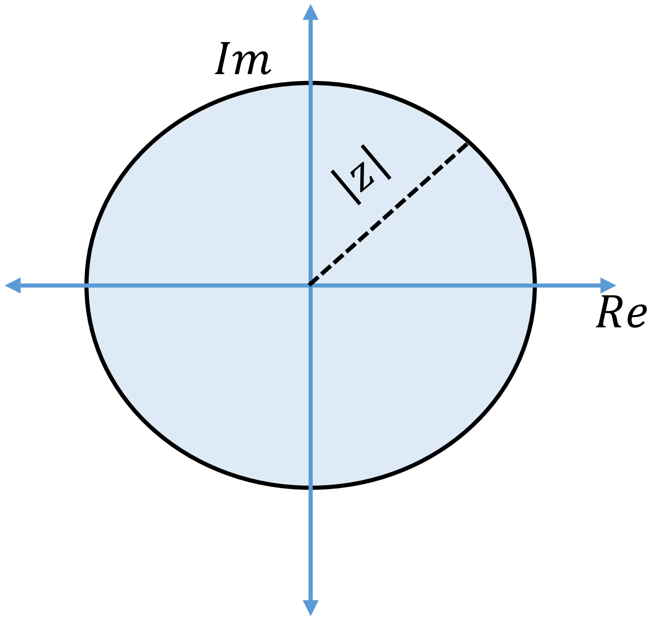
Z-plane

Suppose $\delta = 1$, neighborhood of a point x_0 is the set of all points x such that $|x - x_0| < \delta$ (i.e. $|x - 10|) < 1$. Then neighbourhood are: …9.8, 9.9, 10.1, 10.2…etc.

Suppose $\delta = 1$, neighborhood of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ (i.e. $|z - z_0|) < 1$.

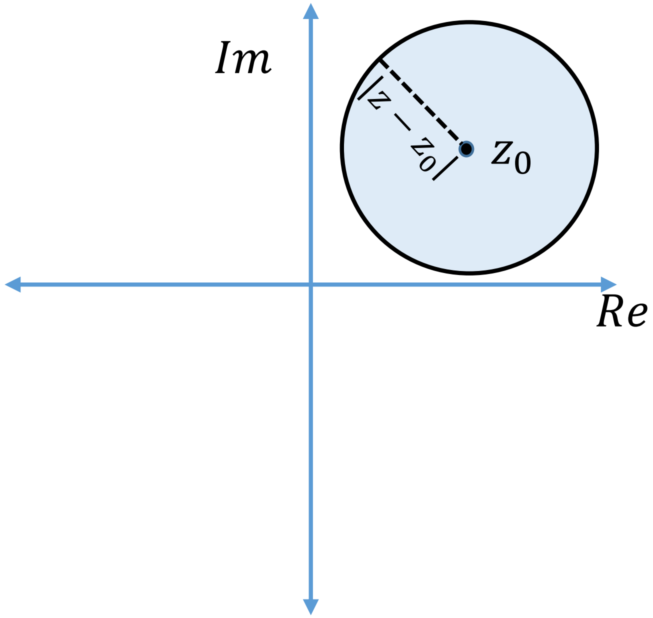
Deleted neighborhood, when we omitted the point z_0 i.e. $0 < |z - z_0| < \delta$.

Identify the region $|z| < 2$



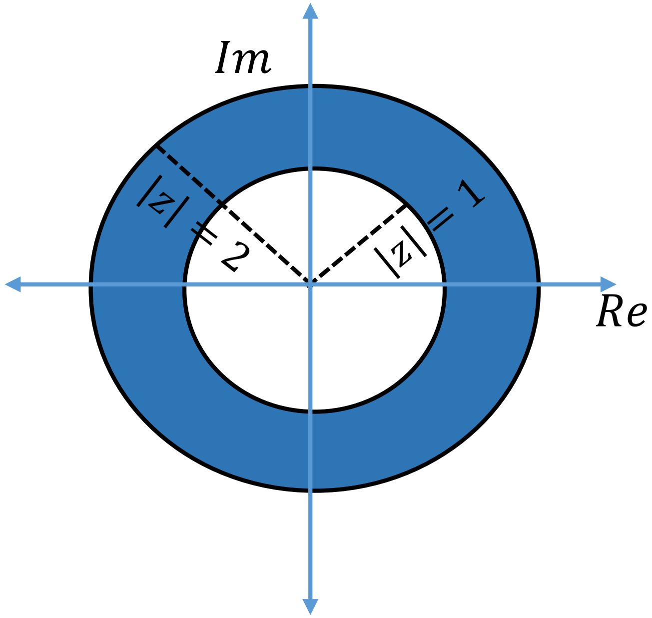
Z-plane

Identify the region $|z - z_0| < 1$



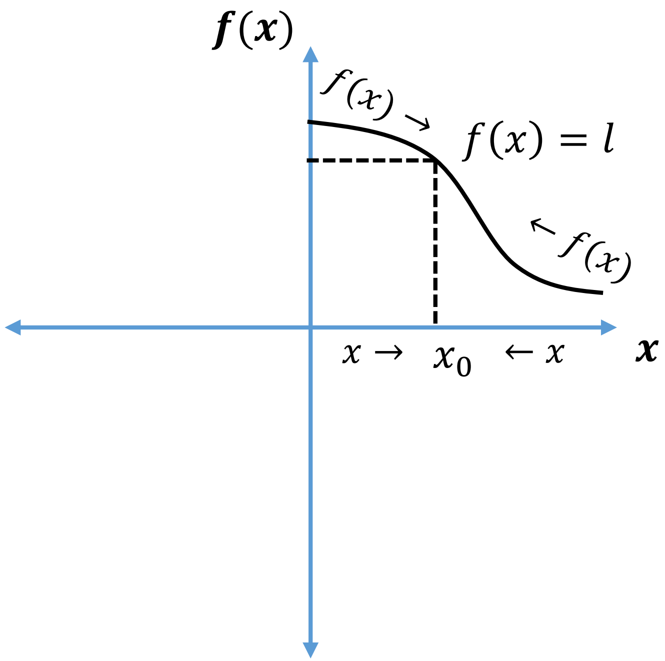
Z-plane

Identify the region $1 < |z| < 2$



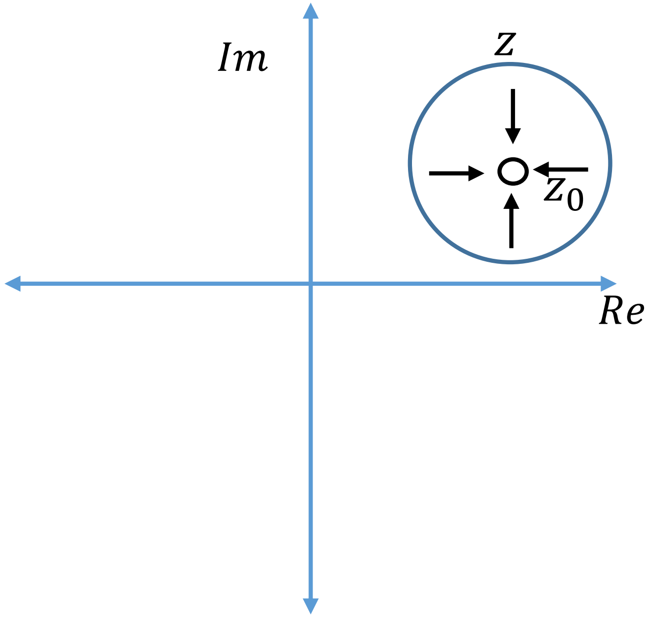
Z-plane

<u>Limit:</u> Let $f\ (z)$ be defined and single-valued in a neighborhood of $z = z_0$ with the possible exception of $z - z_0$ itself (*i.e., in a deleted* δ *neighborhood of* z_0). We say that the number l is the limit of $f\ (z)$ as z approaches z_0 and write $\boldsymbol{lim_{z \to z_0} f(z) = l}$ if for any positive number ϵ (*however small*), we can find some positive number δ (*usually depending on* ϵ) such that $f(z) - l < \epsilon$ whenever $0 < |z - z_0| < \delta$.



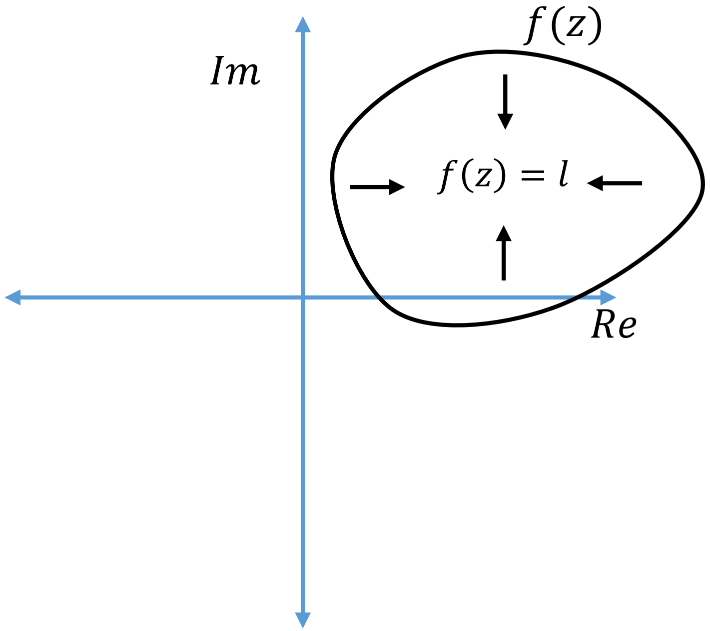
Real function

Here $f(x) - l < \epsilon$
and $0 < |x - x_0| < \delta$



Z-plane

Here $0 < |z - z_0| < \delta$



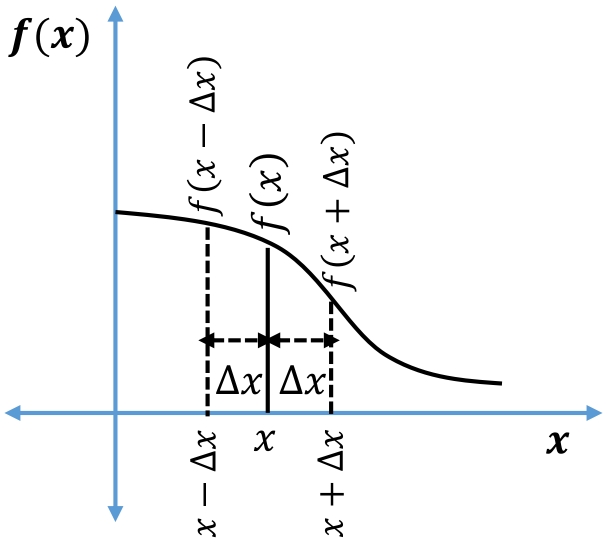
W-plane

Here $f(z) - l < \epsilon$

Derivative: If $f\ (z)$ is single-valued in some region R of the z plane, the derivative of $f\ (z)$ is defined as

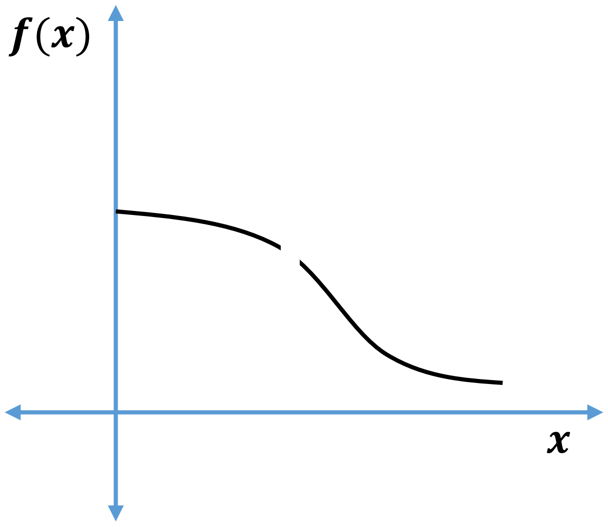
$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f\ (z)$ is differentiable at z. *Although differentiability implies continuity, the reverse is not true.*



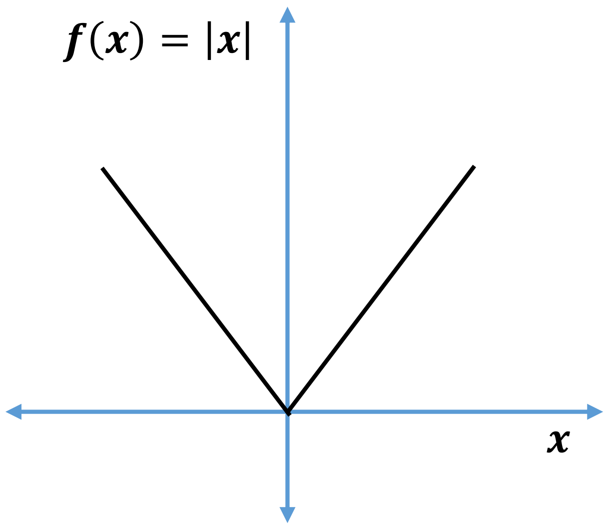
Real function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$



Real function

A discontinuous function can not be differentiable.

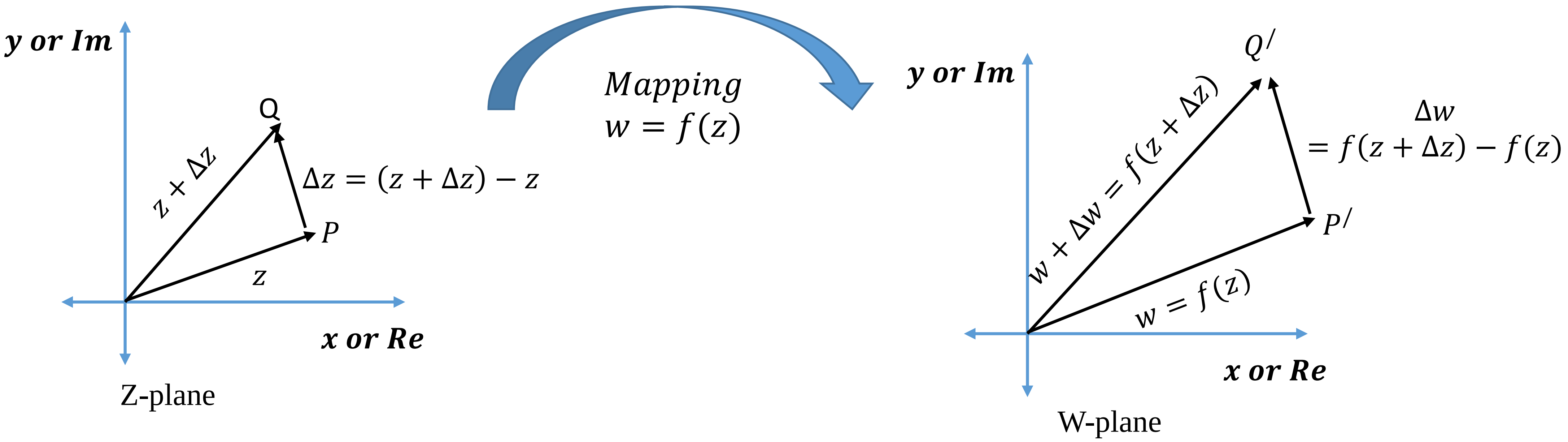


Continuous at the corner but not differentiable.

- Let z be a point P in the z −plane and let w be its image $P^/$ in the w-plane under the transformation $w = f(z)$.

- If we give z an increment Δz, we obtain the point Q of z −plane. This point has image $Q^/$ in the w plane. Thus, from w plane, we see that $P^/Q^/$ represents the complex number $\Delta w = f(z+\Delta z) - f(z)$. It follows that the derivative at z is given

$$f^/(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{Q \to P} \frac{P^/Q^/}{PQ}$$



❖ **Definition:**

- If the derivative $f^{/}(z)$ exists at all points z of a region $\mathcal{R}$, then $f\ (z)$ is said to be analytic in $\mathcal{R}$ and is referred to as an analytic function in $\mathcal{R}$ or a function analytic in $\mathcal{R}$.
- A function $f\ (z)$ is said to be analytic at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f^{/}(z)$ exists.

Q. How do you check the differentiability of a function?

Ans. Checked the limit and continuity of the function at the given region.

Limit → Continuity → Differentiable

Or

Cauchy–Riemann Equations

❖ **Definition:**

Necessary condition

- A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient conditions

- If the partial derivatives of the above equations are continuous in $\mathcal{R}$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$.

- In polar form C-R equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Q. Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e., $f(z) = z^{/}$ is non-analytic anywhere. (*without using C-R equations*)

Solution: By definition,

$$\frac{d}{dz}f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

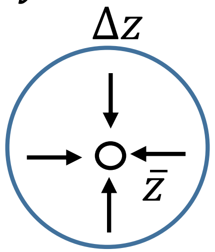
if this limit exists independent of the manner in which $\Delta z = \Delta x + i\Delta y$ approaches zero

$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

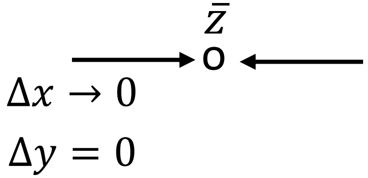
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{(x+iy+\Delta x+i\Delta y)} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x-iy+\Delta x-i\Delta y) - (x-iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



If $\Delta y = 0$, the required limit is

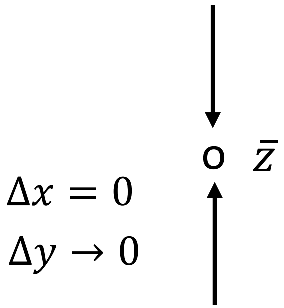


$$\frac{d}{dz}\bar{z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x}{\Delta x}$$

$$= 1$$

If $\Delta x = 0$, the required limit is



$$\frac{d}{dz}\bar{z} = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e., $f(z) = z^{/}$ is non-analytic anywhere.

Cauchy–Riemann Equations (page 77)

Q. Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e., $f(z) = z^{/}$ is non-analytic anywhere. (***using C-R equations***)

Solution: The given function

$$w = f(z) = \overline{x + iy}$$

$$\Rightarrow u(x, y) + iv(x, y) = x - iy$$

$$\therefore u(x, y) = x \quad and \quad v(x, y) = -y$$

Now taking the partial derivative of $f(z)$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial v}{\partial y} = -1$$

$$\left.\right\} \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0$$

$$\left.\right\} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since, the given function does not follow the C-R equations, therefore the function $f(z) = z^{/}$ is non-analytic.

Q. Prove that a (a) necessary and (b) sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region $\mathcal{R}$ is that the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied in $\mathcal{R}$ where it is supposed that these partial derivatives are continuous in R.

$$z = x + iy$$
$$\Delta z = \Delta x + i\Delta y$$
$$w = f(z) = u + iv$$
$$w = f(z) = u(x,y) + iv(x,y)$$
$$w = f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)$$

Solution:

(a) Necessity: In order for $w = f(z)$ to be analytic, the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f^{/}(z)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \qquad \text{-----------(1)}$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero. We consider two possible approaches.

Case I: $\Delta x \to 0, \Delta y = 0$. In this case eq. 1 becomes

$$f^{/}(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\{u(x + \Delta x, y) + iv(x + \Delta x, y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \left\{\frac{u(x + \Delta x, y) - u(x,y)}{\Delta x} + i\left[\frac{v(x + \Delta x, y) - v(x,y)}{\Delta x}\right]\right\} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{-----------(2)}$$

Case II: $\Delta x = 0, \Delta y \to 0$. In this case eq. 1 becomes

$$f^{/}(z) = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\{u(x, y + \Delta y) + iv(x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{i\Delta y}$$

$$= \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left\{ \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right\}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{-----------(3)}$$

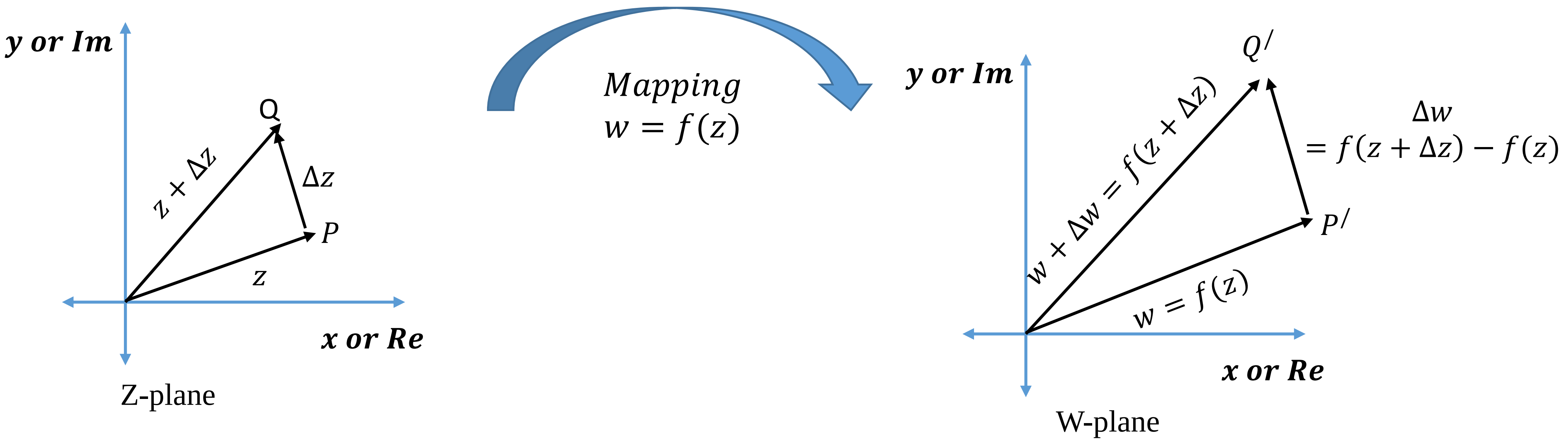
Now $f\,(z)$ cannot possibly be analytic unless these two limits are identical i.e. eq. (2) and (3) are equal. Thus, a necessary condition that $f\,(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$Or$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad and \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad \Bigg\} \quad C - R \ equations$$

Cauchy–Riemann Equations (page 77)



$Since,$ Δw is the small increment of the function $w = f(z)$

$$\Delta w = f(z + \Delta z) - f(z)$$

If $f\ (z)$ is continuous and has a continuous first derivative in a region, then

$$\Delta w = \frac{f(z + \Delta z) - f(z)}{\Delta z}\ \Delta z$$

$$= \{f^{/}(z) + \epsilon\}\ \Delta z$$

$Where\ \epsilon \to 0\ as\ \Delta z \to 0. Now\ we\ can\ write\ the\ differential\ of\ w\ or\ f(z)$

$$dw = f^{/}(z)dz$$

$$f^{/}(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \neq \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f^{/}(z) + \epsilon = \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$When\ \Delta z \to 0\ we\ can\ write\ \Delta z = dz\ and\ \Delta w = dw$

(a) *Sufficiency*: Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are supposed to be continuous, we have

If the partial derivatives of the above equations are continuous in $\mathcal{R}$, then the **Cauchy–Riemann equations** are **sufficient conditions** that $f\,(z)$ be analytic in $\mathcal{R}$.

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}$$

$$= \left\{\frac{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)}{\Delta x}\right\}\Delta x + \left\{\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}\right\}\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + (\frac{\partial u}{\partial y} + \eta_1)\Delta y$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y$$

$Where\ \ \epsilon_1 \to 0\ and\ \ \eta_1 \to 0\ as\ \Delta x \to 0\ and\ \Delta y \to 0.$

Similarly, since $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are supposed to be continuous, we have

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2 \Delta x + \eta_2 \Delta y$$

$Where\ \ \epsilon_2 \to 0\ and\ \ \eta_2 \to 0\ as\ \Delta x \to 0\ and\ \Delta y \to 0.$

Since our function is

$$w = f(z)$$
$$= u + iv$$

Now,

$$\Delta w = \Delta u + i\Delta v$$
$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\Delta x + \eta_1\Delta y + i\frac{\partial v}{\partial x}\Delta x + i\frac{\partial v}{\partial y}\Delta y + i\epsilon_2\Delta x + i\eta_2\Delta y$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + (\epsilon_1 + i\epsilon_2)\Delta x + (\eta_1 + i\eta_2)\Delta y$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \qquad \text{-----------(4)}$$

$Where\ \ \epsilon_1 + i\epsilon_2 \to 0\ and\ \eta_1 + i\eta_2 \to 0\ as\ \Delta x \to 0\ and\ \Delta y \to 0.$

By the Cauchy-Riemann equations, (4) can be written as

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + \frac{\partial v}{\partial x}(i\Delta x - \Delta y) + \epsilon\Delta x + \eta\Delta y$$
$$= \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + \frac{\partial v}{\partial x}i\left(\Delta x - \frac{1}{i}\Delta y\right) + \epsilon\Delta x + \eta\Delta y$$

$$\Delta w = \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + \frac{\partial v}{\partial x} i(\Delta x + i\Delta y) + \epsilon \Delta x + \eta \Delta y$$

Then, on dividing by $\Delta x + i\Delta y$ on both side and taking the limit as $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Delta w = f(z + \Delta z) - f(z)$$

$$\frac{dw}{dz} = f^{/}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

So that the derivative exist and unique i.e. $f(z)$ is analytic in $\mathcal{R}$.

When

Then

We will assume that the **derivative are continuous** and proceed with this assumption → We use **C-R equations** → Get the expression $\frac{dw}{dz} = f^{/}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ → The final expression is nothing but the derivative of the given function f(z) → So the function f(z) is analytic

Sufficient (We use C-R equations → So the function f(z) is analytic)

- *Proof the following relation*

1. $sinz = \frac{e^{iz} - e^{-iz}}{2i}$

2. $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

3. $\sin(hz) = \frac{e^{z} - e^{-z}}{2}$

4. $\cos(hz) = \frac{e^{z} + e^{-z}}{2}$

5. $sin^2 z + cos^2 z = 1$

6. $\sin(iz) = isin(hz)$

7. $\cos(iz) = cos(hz)$

8. $\frac{d}{dz}\sin(hz) = \cos(hz)$

9. $\frac{d}{dz}cos(hz) = sin(hz)$

Q. Show that the function $sinz$ is analytic and hence find the derivative $f^{/}(z)$.

Solution: Since the given function is

$$w = f(z)$$

$$u + iv = sinz$$

$$= \sin(x + iy)$$

$$= \sin x \cos(iy) + cosx \sin(iy)$$

$\sin(iy) = i\ sinhy$

$\cos(iy) = coshy$

$$= \sin x \cos hy + i\ cosx \sin hy$$

$$\therefore u = \sin x \cos hy \quad and\ v = cosx \sin hy$$

Now by using C-R equations

$$\frac{\partial u}{\partial x} = cosx.coshy$$

$$\frac{\partial v}{\partial y} = cosx.coshy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = sinx\ sinhy$$

$$\frac{\partial v}{\partial x} = -sinx.sinhy$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since the C-R equation verified so the function is analytic.

Now,

$$w = f(z) \ = u + iv$$

$$\Rightarrow f^{/}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= cosx.coshy - isinx.sinhy$$

$$= cosx.coshy - sinx.(isinhy)$$

$$= cosx.\cos(iy) - sinx.\sin(iy)$$

$$= \cos(x + iy)$$

$$= \cos(z)$$

$$\therefore f^{/}(z) = \cos(z)$$

Q. Show that the function $\ln(z)$ is analytic and hence find the derivative $f^{/}(z)$.

Solution: Since the given function is

$$w = f(z)$$

$$u + iv = lnz$$

$$= \ln(re^{i\theta})$$

$$= \ln(r) + ln(e^{i\theta})$$

$$= \ln(\sqrt{x^2+y^2} + i\,tan^{-1}(y/x)$$

$Z = re^{i\theta}$

$where, \theta = tan^{-1}(y/x)$

$$\therefore u = \frac{1}{2}\ln(x^2+y^2) \quad and \quad v = tan^{-1}(y/x)$$

Now by using C-R equations

$$\frac{\partial u}{\partial x} = x/(x^2+y^2)$$

$$\frac{\partial v}{\partial y} = x/(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = y/(x^2+y^2)$$

$$\frac{\partial v}{\partial x} = -y/(x^2+y^2)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since the C-R equations are verified so the function is analytic.

Now,

$$w = f(z) = u + iv$$

$$\Rightarrow f^{/}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$f^{/}(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

$$= \frac{x}{(x^2+y^2)} - i.\frac{y}{(x^2+y^2)}$$

$$= \frac{x-iy}{(x^2+y^2)}$$

$$= \frac{x-iy}{(x+iy).(x-iy)}$$

$$= \frac{1}{(x+iy)}$$

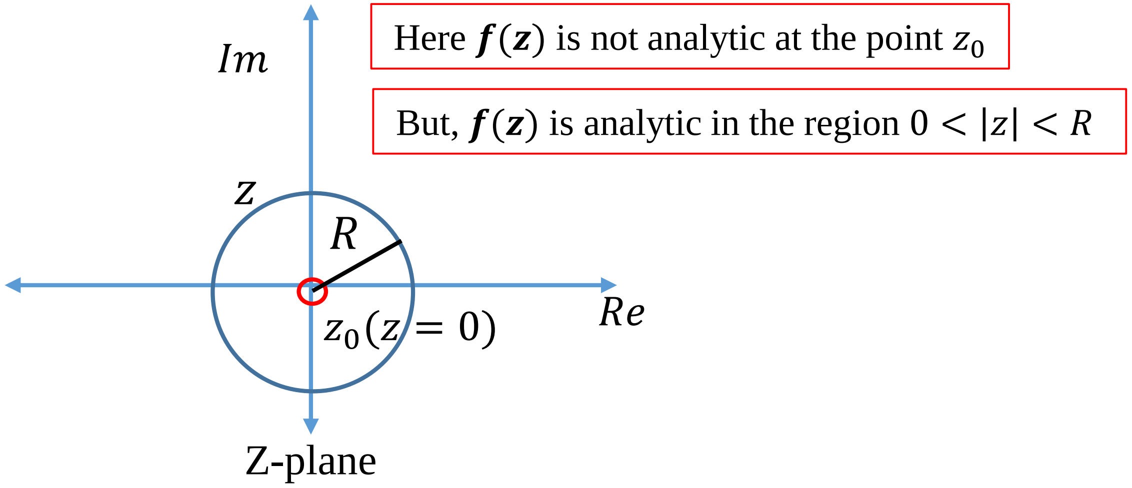
$$\therefore f^{/}(z) = \frac{1}{z}$$

This is the required derivative of $\ln(z)$.

Definition: **Singular point**

If $\boldsymbol{f(z)}$ fails to be analytic in some point $\boldsymbol{Z_0}$ but analytic in some neighbourhood of that point then the point Z_0 is called the singular point or singularity of $f(z)$.

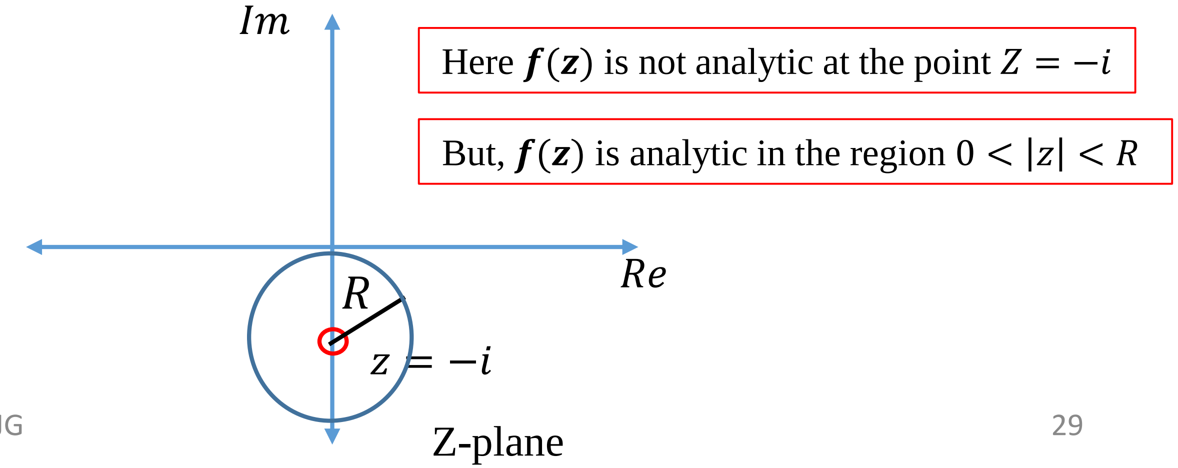
Example 1: Find the singular point of the function $f(z) = \frac{1}{Z}$.



When we put Z=0, the function will blow up and elsewhere the function is analytic except Z=0. So Z=0 is called the singular point.

Example 2: Find the singular point of the function $f(z) = \frac{1}{Z+i}$.

When we put $Z = -i$, the function will blow up and elsewhere the function is analytic except $Z = -i$. So $Z = -i$ is called the singular point.



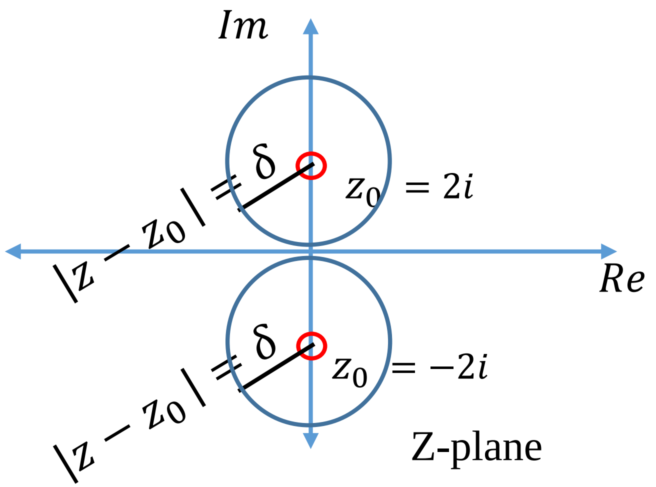
How to find singularity of a given function?

Just put the denominator of the given function equal to zero and solve the equation, the obtained roots or solution of the equation denotes the singularities.

Types of singularities:

1. Isolated singularities: The point $z = z_0$ is called an isolated singularity or isolated singular point of $f\ (z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e., there exists a deleted δ neighborhood of z_0 containing no singularity).

Example 1: Find the singular point of the function $f(z) = \frac{1}{z^2+4}$.



- Here, the singular points are

$$z^2 + 4 = 0$$

$$Or \qquad Z = \pm 2i$$

So, here we get two singular points, one at $Z = 2i$ and another at $Z = -2i$.

- If we can find $\delta > 0$ such that the circle $|z - 2i| = \delta$ encloses no singular point other than $z = 2i$ then this singularity is called isolated singular point.
- If we can find $\delta > 0$ such that the circle $|z + 2i| = \delta$ encloses no singular point other than $z = -2i$ then this singularity is called isolated singular point.

Types of singularities:

2. Poles: If z_0 is an isolated singularity and we can find a positive integer n such that $\lim_{z\to z_0}(z-z_0)^n f(z) = A \neq 0$, then $z - z_0$ is called a pole of order n. If $n = 1$, then z_0 is called a simple pole.

Example 1: Find the singular point of the function $f(z) = \frac{1}{z^2+4}$.

➢ Here, the singular points are

$$z^2 + 4 = 0$$

$$Or \qquad Z = \pm 2i$$

So, here we get two singular points, one at $Z = 2i$ and another at $Z = -2i$. Now, for $Z = 2i$

$$= \lim_{z\to z_0}(z-z_0)^n f(z)$$

$$= \lim_{z\to 2i}(z-2i)^1 \frac{1}{z^2+4}$$

$$= \lim_{z\to 2i}(z-2i)^1 \frac{1}{(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i}\frac{1}{(z+2i)}$$

$$= 1/4i$$

- So $\boldsymbol{Z = 2i}$ **is a pole of order 1 or simple pole.**

Similarly, for $Z = -2i$

$$= \lim_{z\to z_0}(z-z_0)^n f(z)$$

$$= \lim_{z\to -2i}(z+2i)^1 \frac{1}{z^2+4}$$

$$= \lim_{z\to -2i}(z+2i)^1 \frac{1}{(z+2i)(z-2i)}$$

$$= \lim_{z\to -2i}\frac{1}{(z-2i)}$$

$$= 1/-4i$$

- So $\boldsymbol{Z = -2i}$ **is a pole of order 1 or simple pole.**

Example 2: Find the singular point of the function $f(z) = \frac{1}{(z^2+4)^2}$.

- Here, the singular points are

$$z^2 + 4 = 0$$

$$Or \qquad Z = \pm 2i$$

So, here we get two singular points, one at $Z = 2i$ and another at $Z = -2i$. Now, for $Z = 2i$

$$= \lim_{z \to z_0} (z - z_0)^n f(z)$$

$$= \lim_{z \to 2i} (z - 2i)^2 \frac{1}{(z^2 + 4)^2}$$

$$= \lim_{z \to 2i} (z - 2i)^2 \frac{1}{(z - 2i)^2 (z + 2i)^2}$$

$$= \lim_{z \to 2i} \frac{1}{(z + 2i)^2}$$

$$= 1/-16$$

- So $\boldsymbol{Z = 2i}$ **is a pole of order 2.**

Similarly, for $Z = -2i$

$$= \lim_{z \to -2i} (z - 2i)^2 \frac{1}{(z^2 + 4)^2}$$

$$= \lim_{z \to -2i} (z - 2i)^2 \frac{1}{(z - 2i)^2 (z + 2i)^2}$$

$$= \lim_{z \to -2i} \frac{1}{(z - 2i)^2}$$

$$= 1/-16$$

- So $\boldsymbol{Z = -2i}$ **is a pole of order 2.**

Example 3: $f(z) = \frac{1}{(z-3)^2}$ has a pole of order 2 at $z = 3$.

Example 4: $f(z) = \frac{(3z-2)}{(z-2)^2(z-1)}$ has a pole of order 2 at $z = 2$, and simple pole at $z = 1$.

Types of singularities:

3. Branch point: Branch Points of multiple-valued functions are **non-isolated singular points** since a multiple-valued function is not continuous and, therefore, not analytic in a deleted neighborhood of a branch point.

Example 1: $f(z) = (z-3)^{1/2}$ has a branch point at $z = 3$. This branch point i.e. $z = 3$ is called a non-isolated singular point.

Example 2: what is the singular point of $f(z) = (z-3)$.

- It does not has any singular point.

Example 3: $f(z) = \ln(z^2 + z - 2)$ has a branch point where $z^2 + z - 2 = 0, \text{i.e.}, \text{at } z = 1 \text{ and } z = -2.$ These branch points are called as non–isolated singular point.

Types of singularities:

4. Removable singularities: An isolated singular point z_0 is called a removable singularity of $f\ (z)$ if $\lim_{z \to z_0} f(z)$ exists. By defining $f(z_0) = \lim_{z \to z_0} f(z)$, it can then be shown that $f\ (z)$ is not only continuous at z_0 but is also analytic at z_0.

Example 1: Find the type of singular point of $f(z) = \frac{sinz}{z}$

- $In\ the\ given\ function,\ z = \mathbf{0}\ is\ singular\ point.\ Since\ if\ we\ put\ z{=}0\ then\ the\ given\ function\ will\ blow\ up.$

$Now,$

$$f(z) = \frac{sinz}{z}$$

$$f(z) = \frac{1}{z}(sinz)$$

$$f(z) = \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots)$$

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

Hence we have seen that the singular point $\boldsymbol{z = 0}$ is removable.

Again,

$$\lim_{z \to 0} f(z) = \lim_{z \to 0}(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots)$$

$$= 1$$

It has been found that $f\ (z)$ if $\lim_{z \to 0} f(z)$ exists

Therefore, the given function has a removable singular point at $z = 0$.

Types of singularities:

5. Essential singularities: An isolated singularity that is not a pole or removable singularity is called an essential singularity.

Example 1: Find the type of singular point of $f(z) = e^{1/z}$

- Let us expand the given function

$$f(z) = e^{1/z}$$

$$= 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

So, it is found that $z = 0$ is a singular point and it is neither removable singularity nor a pole (since the power is goes on increasing).

Such type of singularities are called ***essential singularities***.

If a function has an isolated singularity, then the singularity is either ***removable***, ***a pole***, or ***an essential singularity***.

Laurent Series

Definition:

- In mathematics, an integral assigns numbers to functions in a way that describes area, volume, and other concepts that arise by combining infinitesimal data. The process of finding integrals is called integration. *(from Wikipedia)*

Integration is basically summation, but with some differences

- Summation has been used when the data are discrete
- Integration has been used when the data are continuous

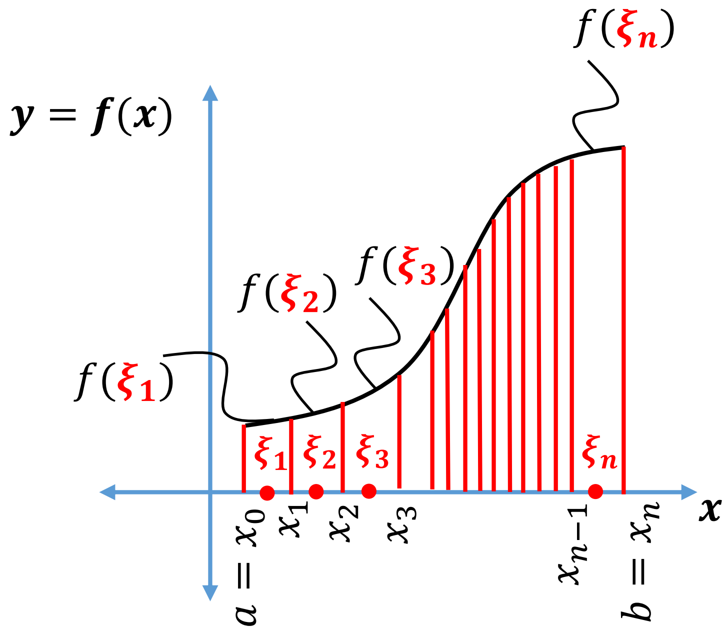
$$Area(under\ the\ curve)$$

$$= Add\ up\ all\ the\ area\ of\ rectangular\ strip$$

$$= f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1})$$

$$= \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$



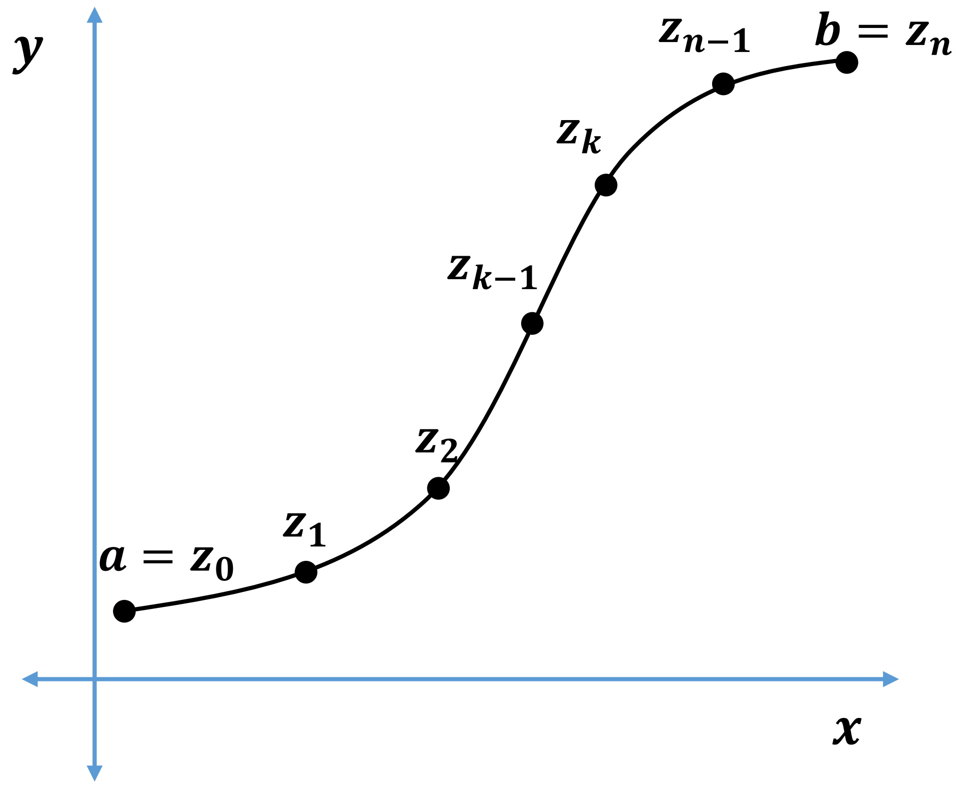
When $\Delta x_k \to 0$, (*exist only when the function is continuous*)

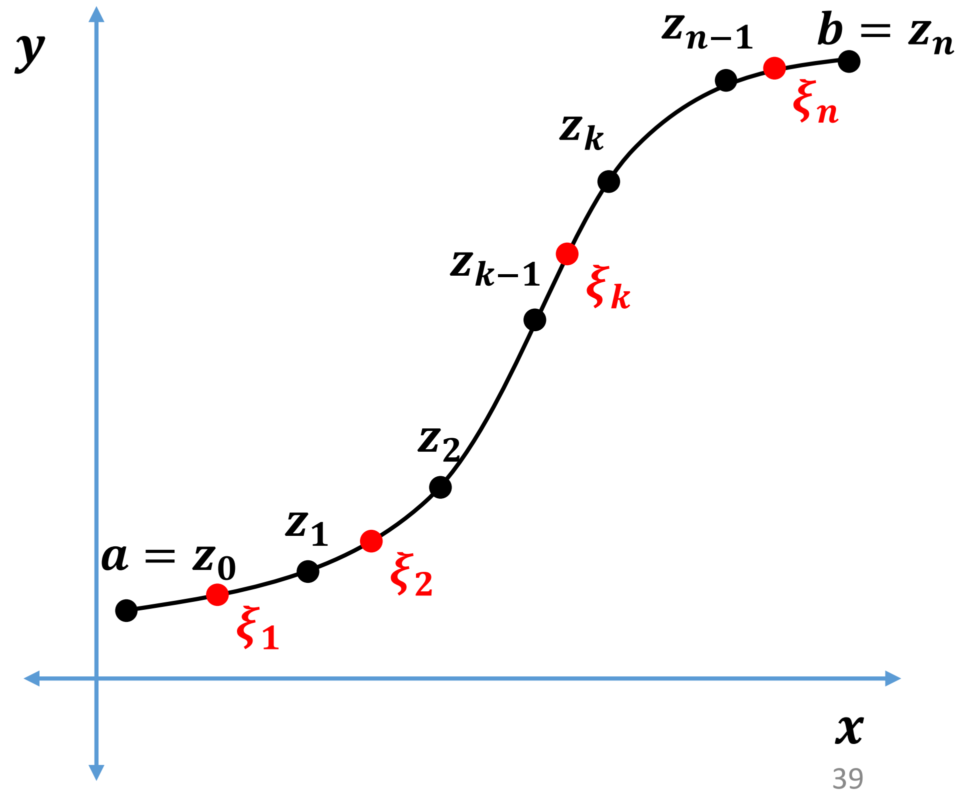
$$Area(under\ the\ curve) = \int_a^b f(x)dx$$

- Let $f\ (z)$ be continuous at all points of a curve C Fig., which we shall assume has a finite length, i.e., C is a rectifiable curve.

Now, subdivide C into n parts by means of points $\boldsymbol{z_1, z_2, \ldots, z_{n-1}}$, chosen arbitrarily, and call $\boldsymbol{a = z_0}$, $\boldsymbol{b = z_n}$.

On each arc joining $\boldsymbol{z_{k-1}}$ to $\boldsymbol{z_k}$ [where k goes from 1 $to\ n$], choose a point ξ_k.





- Let $f\ (z)$ be continuous at all points of a curve C Fig., which we shall assume has a finite length, i.e., C is a rectifiable curve.

Now, subdivide C into n parts by means of points $\boldsymbol{z_1, z_2, \ldots, z_{n-1}}$, chosen arbitrarily, and call $\boldsymbol{a = z_0}$, $\boldsymbol{b = z_n}$.

On each arc joining $\boldsymbol{z_{k-1}}$ to $\boldsymbol{z_k}$ [where k goes from 1 to n], choose a point ξ_k.

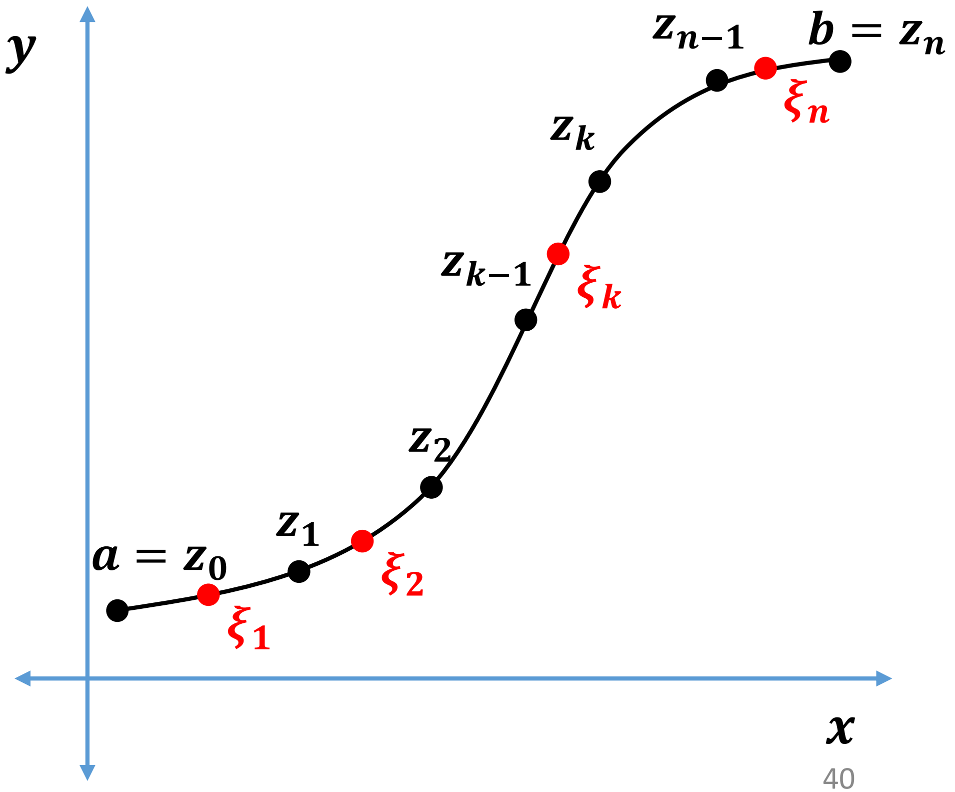
From the sum,

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1})$$

On writing $\boldsymbol{z_k - z_{k-1} = \Delta z_k}$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$

$$= \sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

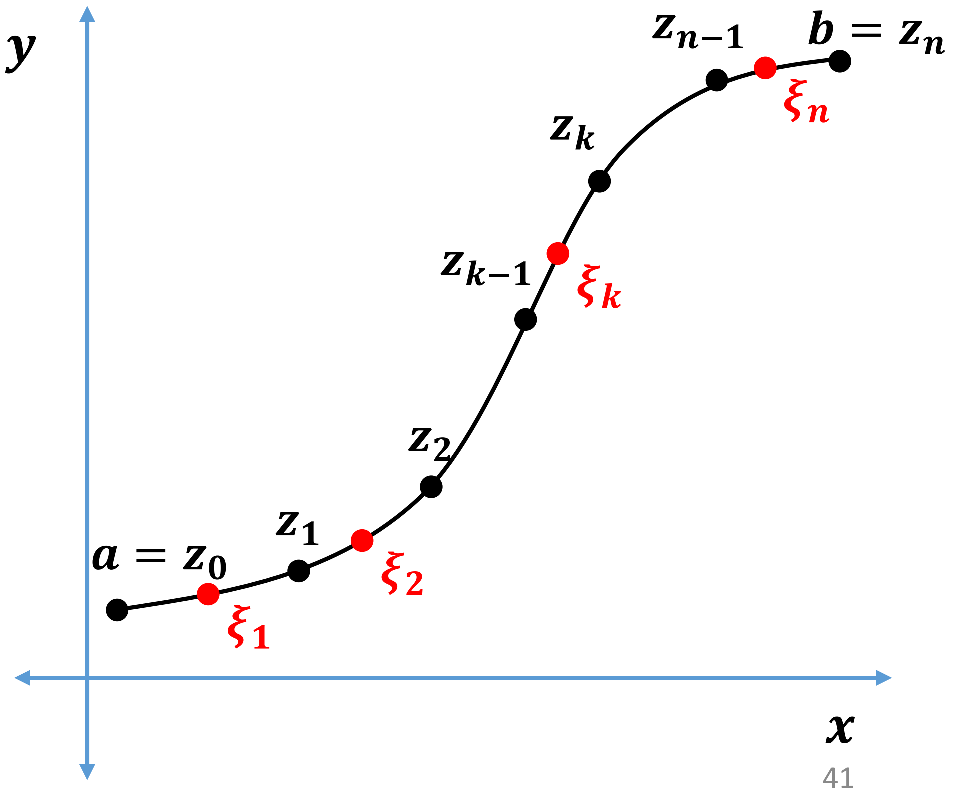


Let the number of subdivisions $\boldsymbol{n}$ increase in such a way that the largest of the chord lengths $|\boldsymbol{\Delta z_k}|$ approaches zero. Then, since $f\,(z)$ is continuous, the sum S_n approaches a limit that does not depend on the mode of subdivision and we denote this limit by

$$\int_{a}^{b} f(z)dz \quad or \quad \int_{C} f(z)dz$$

called the **complex line integral** or **simply line integral** of $f\,(z)$ along curve C, or the definite integral of $f\,(z)$ from a *to* b along curve C.

In such a case, $f\,(z)$ is said to be integrable along C. If $f\,(z)$ is **analytic** at all points of a region R and if C is a curve lying in R, then $f\,(z)$ is continuous and therefore integrable along C.



Q. Calculate the line integral of the function $\vec{\boldsymbol{v}} = \boldsymbol{y^2\hat{x} + 2x(y+1)\hat{y}}$ from the point $\boldsymbol{a} = (\mathbf{1,1,0})$ to the point $\boldsymbol{b} = (\mathbf{2,2,0})$ along path along the path (1) *and* (2) as shown in figure. Also find the close path integral that goes from a to b along $path$ (1) and return to a along $path$ (2).

Solution: Since, we know

$$\vec{dl} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

Path (1) consist of two parts.
Along the **(i) horizontal segment**, $dy = dz = 0, so$

$$\vec{dl} = dx\hat{x}, \quad y = 1,$$

$$\therefore \vec{v}.\vec{dl} = y^2dx \;= dx$$

So, the line integral

$$\int_1^2 \vec{v}.\vec{dl} = \int_1^2 dx \;= 1$$

On the **(ii) vertical segment**, $dx = dz = 0, so$

$$\vec{dl} = dy\hat{y}, \quad x = 2,$$

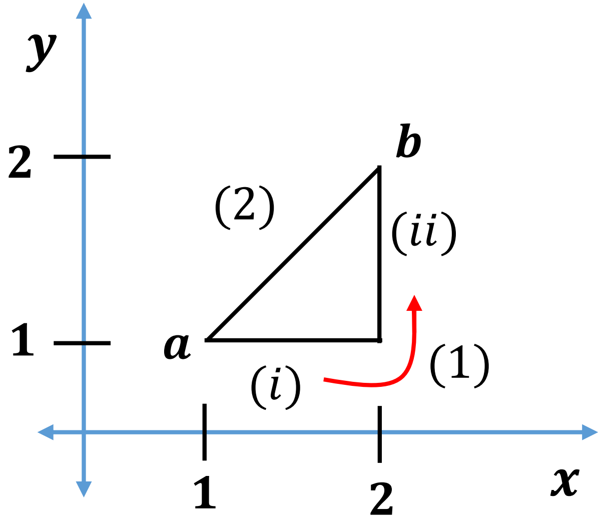
$$\therefore \vec{v}.\vec{dl} = 4(y+1)dy$$

So, the line integral

$$\int_1^2 \vec{v}.\vec{dl} = \int_1^2 4(y+1)dy \;= 10$$

So, by the path (1)

$$\int_a^b \vec{\boldsymbol{v}}.\vec{\boldsymbol{dl}} = \mathbf{1 + 10 = 11}$$



Meanwhile, on path (2), $x = y, dx = dy, and\ dz = 0, so$

$$\vec{dl} = dx\hat{x} + dx\hat{y},$$

$$\therefore \vec{v}.\vec{dl} = x^2dx + 2x(x+1)dx = (3x^2 + 2x)dx$$

So, the line integral along path (2)

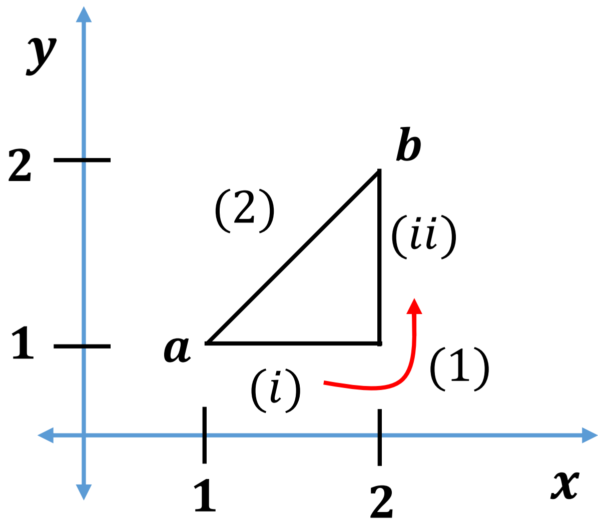
$$\int_a^b \vec{\boldsymbol{v}}.\vec{\boldsymbol{dl}} = \int_1^2 (\mathbf{3x^2 + 2x})\boldsymbol{dx} \;= \mathbf{10}$$

Now, for the loop that goes out (1) and back (2)

$$\oint \vec{v}.\overrightarrow{dl} = 11 - 10 = 1$$

Also, for the loop that goes out (2) and back (1)

$$\oint \vec{v}.\overrightarrow{dl} = 10 - 11 = -1$$

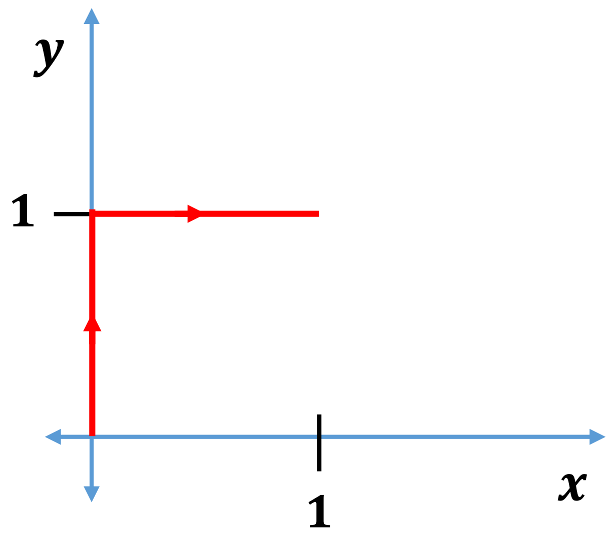


So, what we have found in this line integration

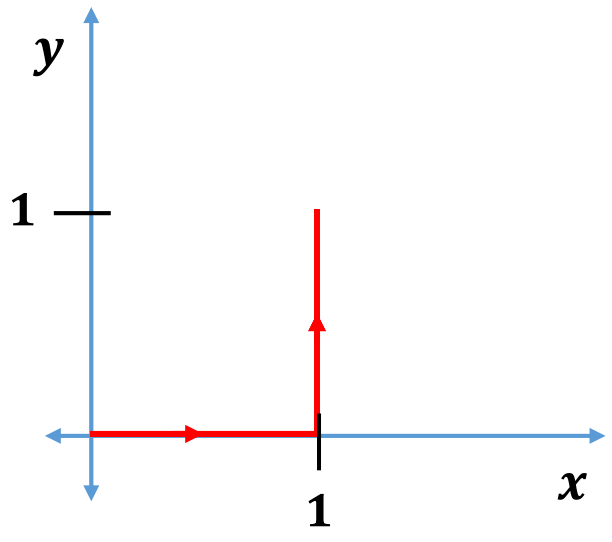
- The line integration along path (1) and path (2) both have different values.
- The closed line integration for the loop that goes out (1) and back (2) is different for that of goes out (2) and back (1).

The line integration is path dependent for real function

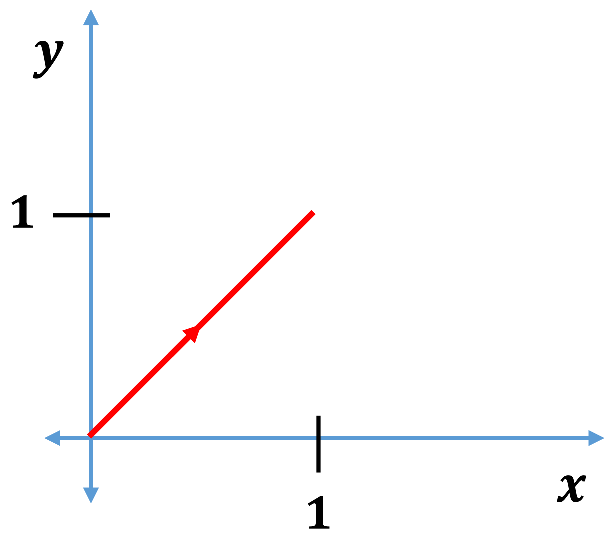
Q. Find the line integral for $f(z) = z$ for the path as shown in figure.



Path (1)



Path (2)



Path (3)

Solution:

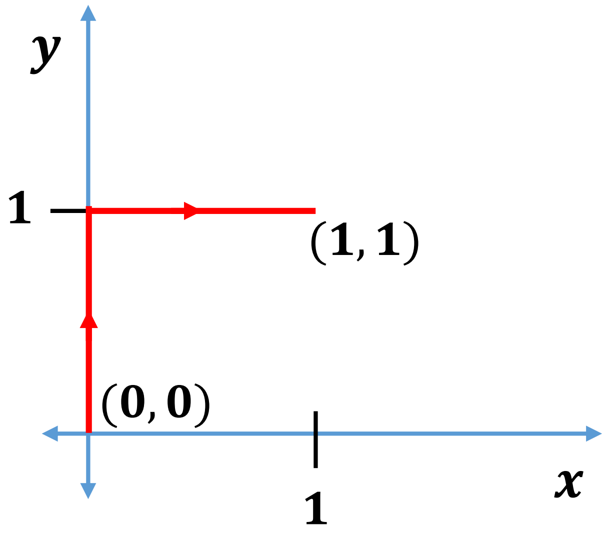
Since,

$$z = x + iy$$

$$\therefore\, dz = dx + idy$$

$$Now, f(z) = x + i\,y$$

$$Therefore, \int_C f(z)dz = \int_c (x + iy)(dx + idy)$$



Path (1)

Path (1):

$$\int_C f(z)dz = \int_0^1 \underbrace{(0 + iy)(0 + idy)}_{x=0,\, y=0\to 1} + \int_0^1 \underbrace{(x + i.1)(dx + i.0)}_{x=0\to 1,\, y=1}$$

$$= \int_0^1 (-ydy) + \int_0^1 (xdx + idx)$$

$$= -\frac{1}{2} + 0 + \frac{1}{2} + i$$

$$= i$$

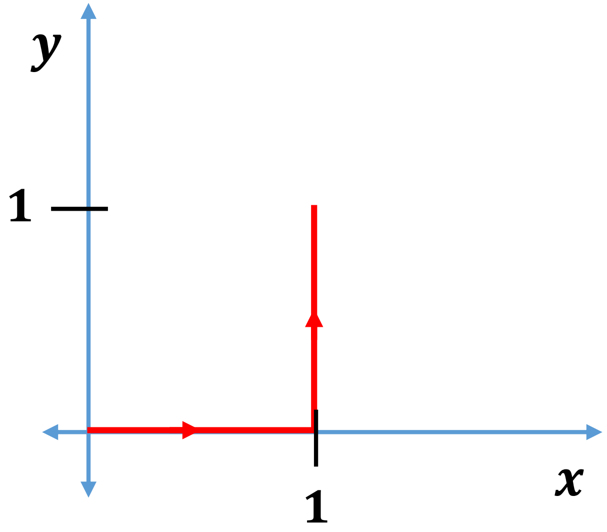
Path (2):

$$\int_C f(z)dz = \int_0^1 \underbrace{(x+0)(dx+0)}_{x=0\to 1,\, y=0} + \int_0^1 \underbrace{(1+i.y)(0+i.dy)}_{x=1,\, y=0\to 1}$$

$$= \int_0^1 (xdx) + \int_0^1 (idy - ydy)$$

$$= \frac{1}{2} + i - \frac{1}{2}$$

$$= i$$



Path (2)

Path (3):

$x = y, dx = dy, x = 0 \to 1$

$$\int_C f(z)dz = \int_0^1 (x + ix)(dx + idx)$$

$$= \int_0^1 (xdx + i.2xdx - xdx)$$

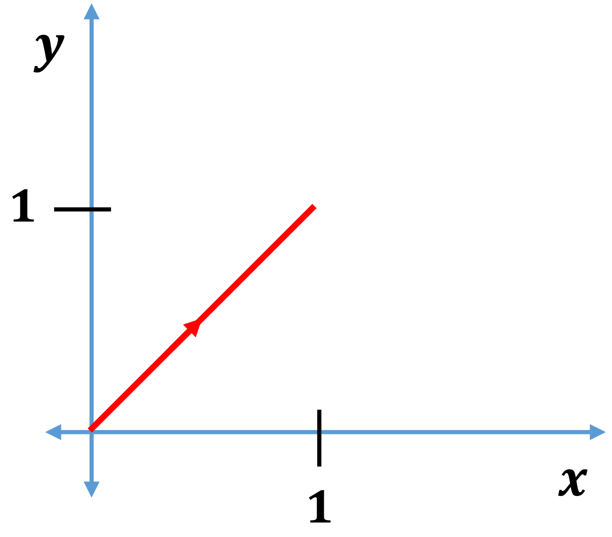
$$= \int_0^1 (i.2xdx)$$

$$= i.2.\frac{1}{2}$$

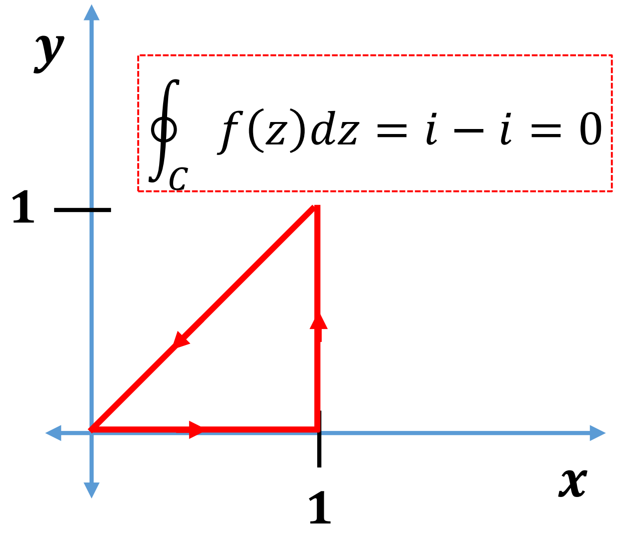
$$= i$$

So, it is found that the line integration of analytic complex function is path independent.

The close path/line integral of analytic complex function is always zero.

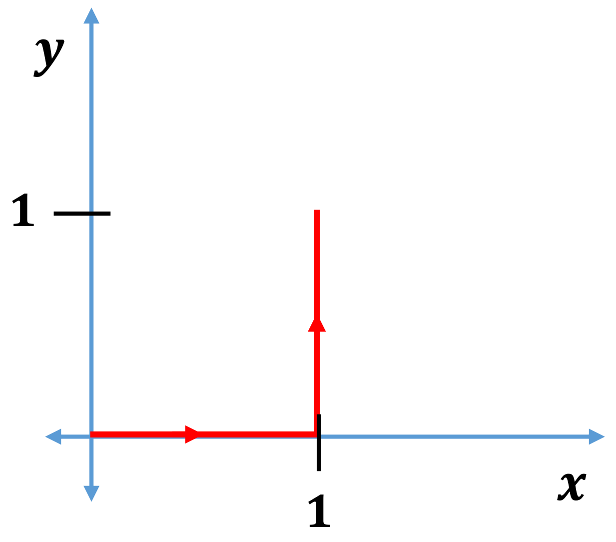


Path (3)

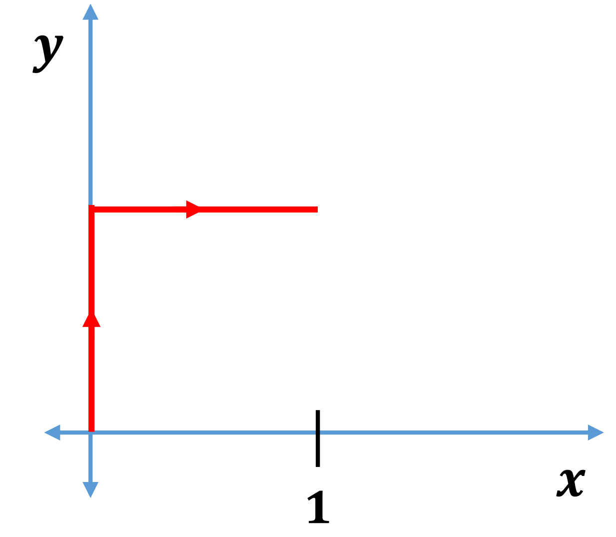


$$\oint_C f(z)dz = i - i = 0$$

Q. Find the line integral for $\boldsymbol{f(z) = \bar{z}}$ for the path as shown in figure.



Path (1)



Path (2)

Solution: Since,

$$z = x + iy \quad \Rightarrow \bar{z} = x - iy$$

$$\therefore dz = dx + idy$$

$$Now, f(z) = x - iy$$

$$Therefore, \int_C f(z)dz = \int_c (x - iy)(dx + idy)$$

Path (1):

$$\int_C f(z)dz = \underbrace{\int_0^1 (x - 0)(dx + 0)}_{x = 0 \to 1,\, y = 0} + \underbrace{\int_0^1 (1 - i.y)(0 + i.dy)}_{x = 1,\, y = 0 \to 1}$$

$$= \int_0^1 (xdx) + \int_0^1 (i.dy + ydy)$$

$$= \frac{1}{2} + i + \frac{1}{2}$$

$$= 1 + i$$

Path (2):

$$\int_C f(z)dz = \int_0^1 \underbrace{(0 - iy)(0 + idy)}_{x = 0,\, y = 0 \to 1} + \int_0^1 \underbrace{(x - i.1)(dx + i.0)}_{x = 0 \to 1,\, y = 1}$$

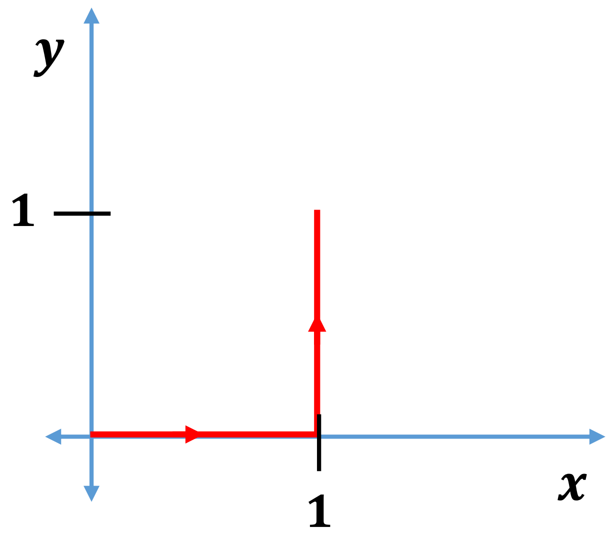
$$= \int_0^1 (ydy) + \int_0^1 (xdx - idx)$$

$$= \frac{1}{2} + \frac{1}{2} - i$$

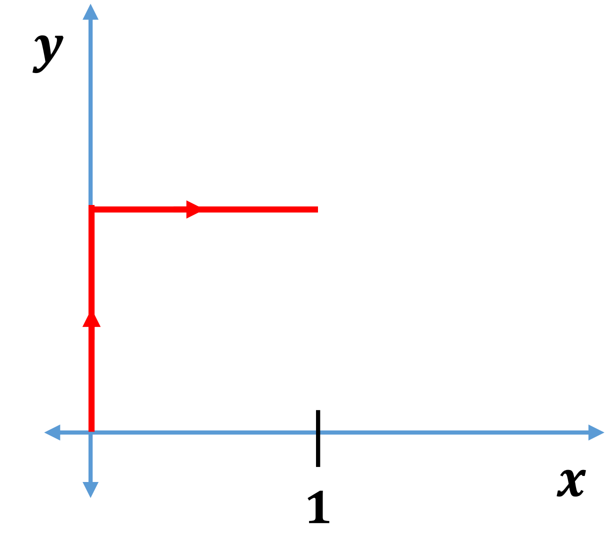
$$= 1 - i$$

So, it is found that the line integration of non-analytic complex function is path dependent.

The close path/line integral of non-analytic complex function is not zero.



Path (1)



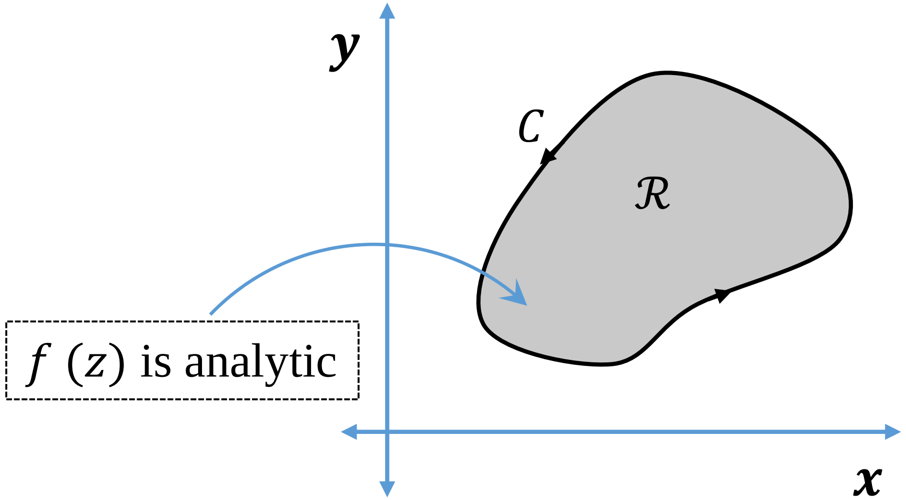
Path (2)

Cauchy's Theorem (Statement):

- Let $f\ (z)$ be analytic in a region $\mathcal{R}$ and on its boundary C. Then

$$\oint_C f(z)dz = 0$$

This fundamental theorem, often called *Cauchy's integral theorem* or *simply Cauchy's theorem* or *Cauchy–Goursat theorem*, is valid for both **simply- and multiply-connected regions**.



Note: This is not Cauchy's integral formula.

Q. What is a simple closed curve? *(page 83)*

➢ A closed curve that does not intersect itself anywhere is called a simple closed curve.

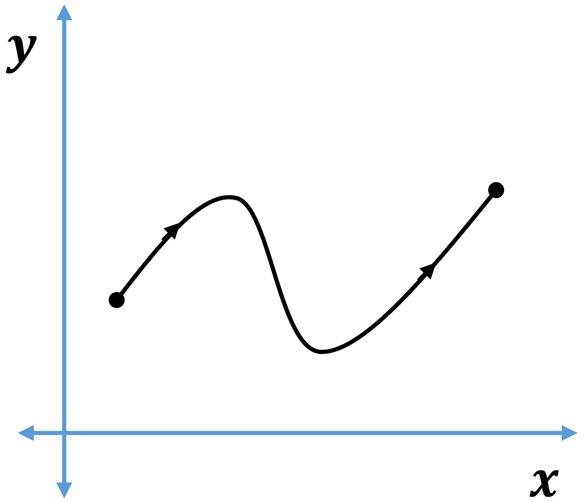


Fig.: Continuous curve or arc

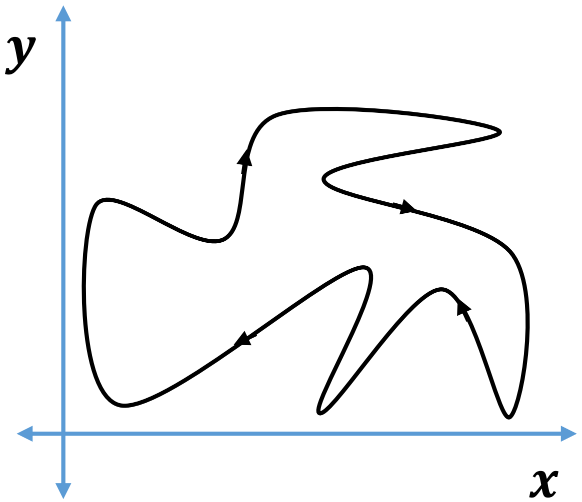


Fig.: Simple closed curve

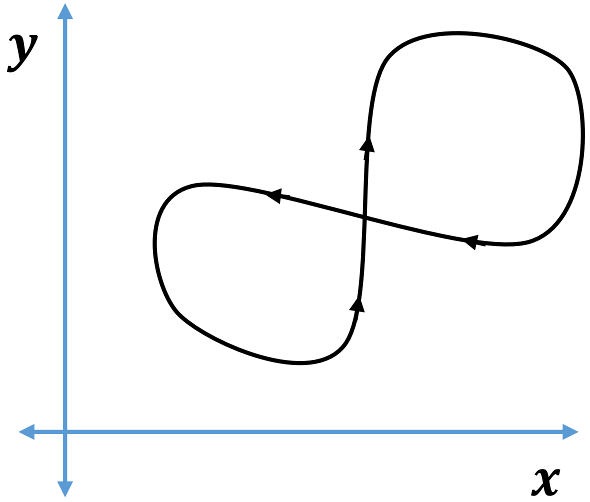


Fig.: Non-Simple closed curve

Simply and Multiply Connected (Page 113)

- A region $\mathcal{R}$ is called simply-connected if any **simple closed curve**, which lies in $\mathcal{R}$, can be **shrunk to a point** without leaving $\mathcal{R}$. A region $\mathcal{R}$, which is not simply-connected, is called multiply-connected.

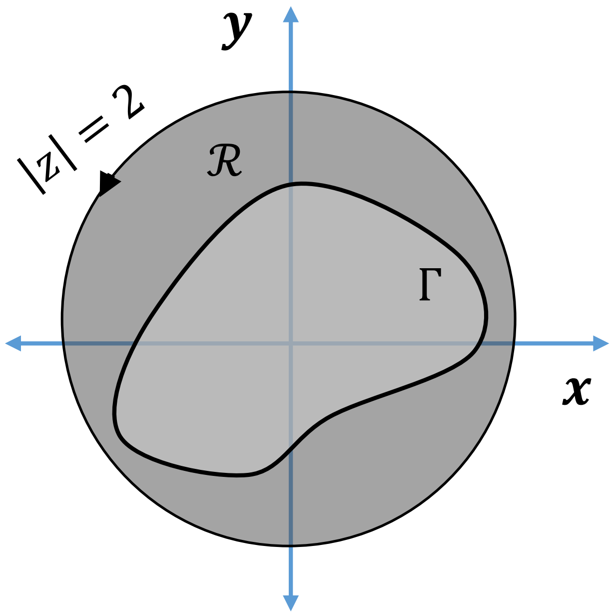


Fig.: (1) simply-connected region

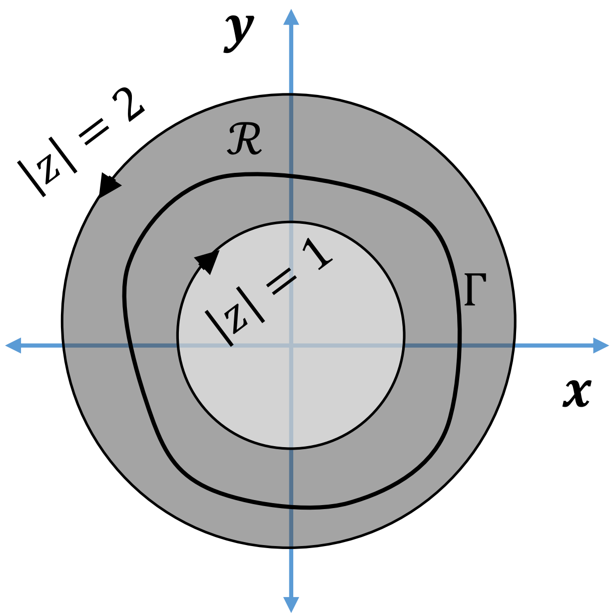


Fig.: (2) multiply-connected region with one hole

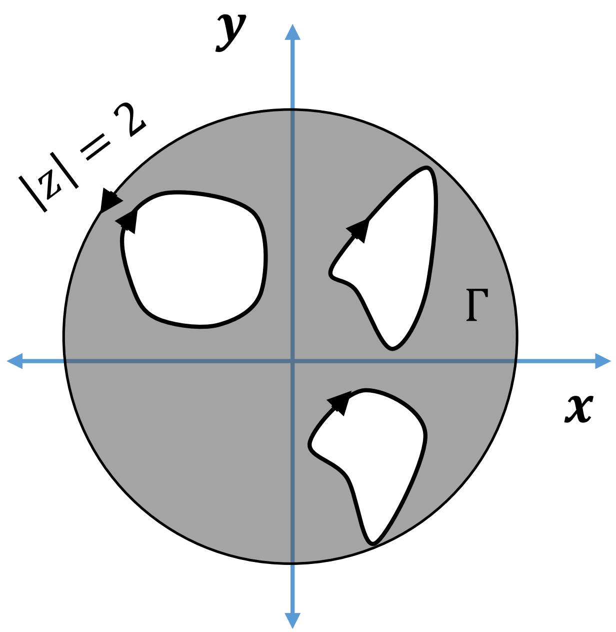


Fig.: (3) multiply-connected region with three holes

Here, $\mathcal{R}$ is the region defined by $|z| < 2$.
Γ is any simple closed curve lying in $\mathcal{R}$ and it can be shrunk to a point that lies in $\mathcal{R}$, and thus does not leave R, so that R is simply-connected

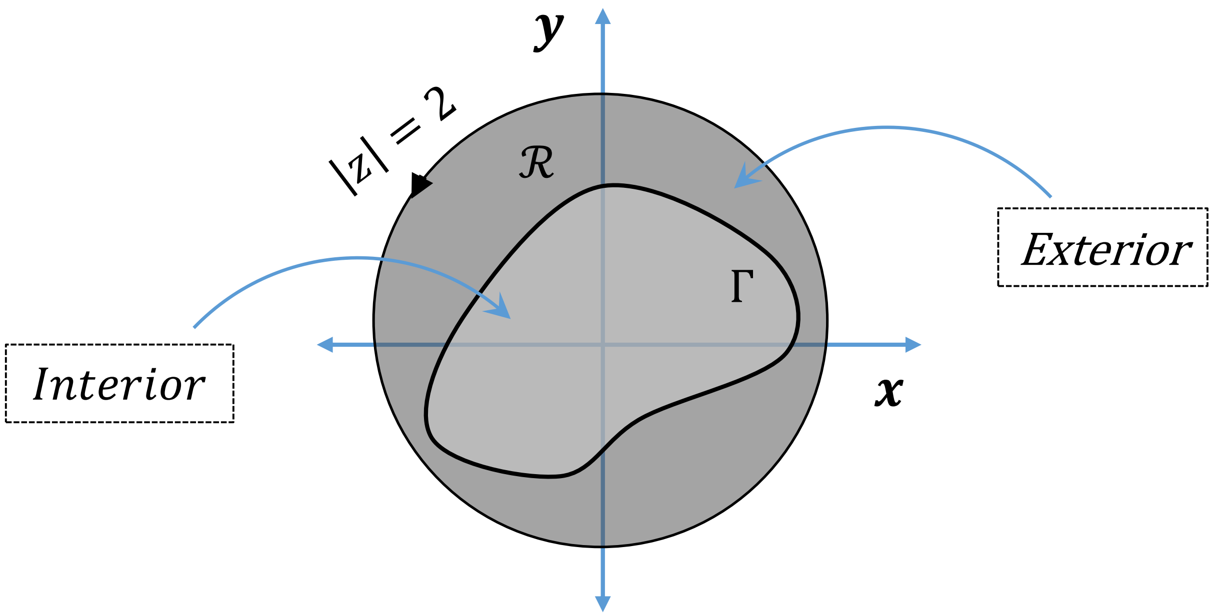
Here, $\mathcal{R}$ is the region defined by $1 < |Z| < 2$.
Γ is any simple closed curve lying in $\mathcal{R}$ and it can not possibly shrunk to a point without leaving $\mathcal{R}$, so that $\mathcal{R}$ is multiply-connected.

Jordan Curve

Any continuous, closed curve that does not intersect itself and may or may not have a finite length, is called a Jordan curve

Jordan Curve Theorem

A Jordan curve divides the plane into two regions having the curve as a common boundary. That region, which is bounded [*i.e., is such that all points of it satisfy* $|z| < M$ *, where M is some positive constant*], is called the **interior or inside** of the curve, while the other region is called the **exterior or outside** of the curve.



- The boundary C of a region is said to be traversed in the **positive sense** or direction if an observer travelling in this direction [*and perpendicular to the plane*] has the region to the left.

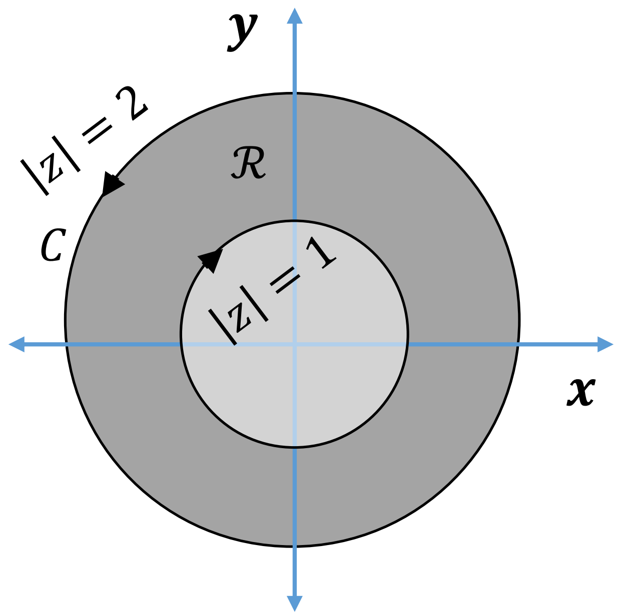
- We use the special symbol

$$\oint_C f(z)dz = 0$$

to denote integration of $f\ (z)$ around the boundary C in the **positive sense**. The integral around C is often called a **contour integral**.

Note:

- In the case *as shown in the figure*, the positive direction is the counterclockwise direction for the **outer circle**.
- In the case *as shown in the figure*, the positive direction is the clockwise direction for the **inner circle**.



Contour:

A curve, which is composed of a finite number of smooth arcs, is called a piecewise or sectionally smooth curve or sometimes a contour.

Or

An outline representing or bounding the shape or form of something.

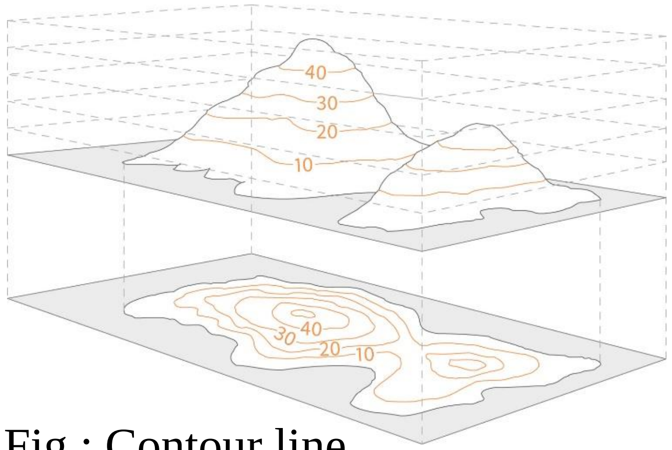


Fig.: Contour line.

Contour Integration:

In the mathematical field of complex analysis, contour integration is a method of evaluating certain integrals along paths in the complex plane.

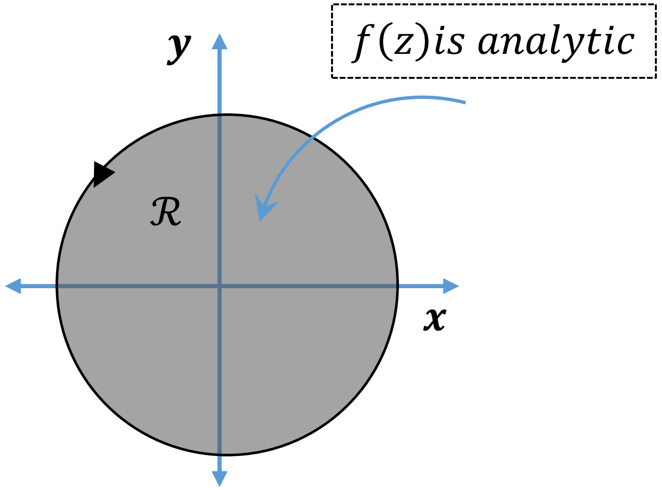
Contour integration methods include:

- direct integration of a **complex-valued function** along a curve in the complex plane (a contour)
- application of the **Cauchy integral formula**; and
- application of the **residue theorem**.

- Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous partial derivatives in a region R and on its boundary C. Green’s theorem states that

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

The theorem is valid for both simply- and multiply-connected regions.

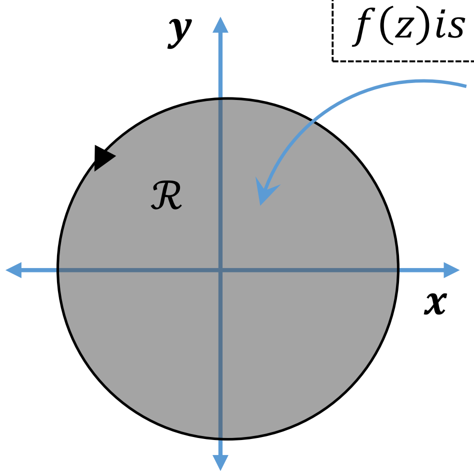


We can simply replace the P and Q with u and v to get a better visualization

- Let $u(x, y)$ and $v(x, y)$ be continuous and have continuous partial derivatives in a region Rand on its boundary C. Green’s theorem states that

$$\oint_C udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

Q. Prove Cauchy's theorem $\oint_C f(z)dz = 0$ if $f(z)$ is analytic and continuous at all points inside and on a simple closed curve C *Or Prove Cauchy's theorem for simply connected region.*



Solution:

Let us consider, $f(z) = u + iv$, where $z = x + iy$ and u and v are function of x and y.

Since $f(z)$ is analytic and has a continuous derivative, so we can write

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

using C − R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The given contour integration can be written as,

$$\oint_C f(z)dz = \oint_C (u + iv)(dx + idy) = \oint_C udx - vdy + i\oint_C vdx + udy$$

Now, using Green's theorem

$$\oint_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy = 0$$

using C − R equations

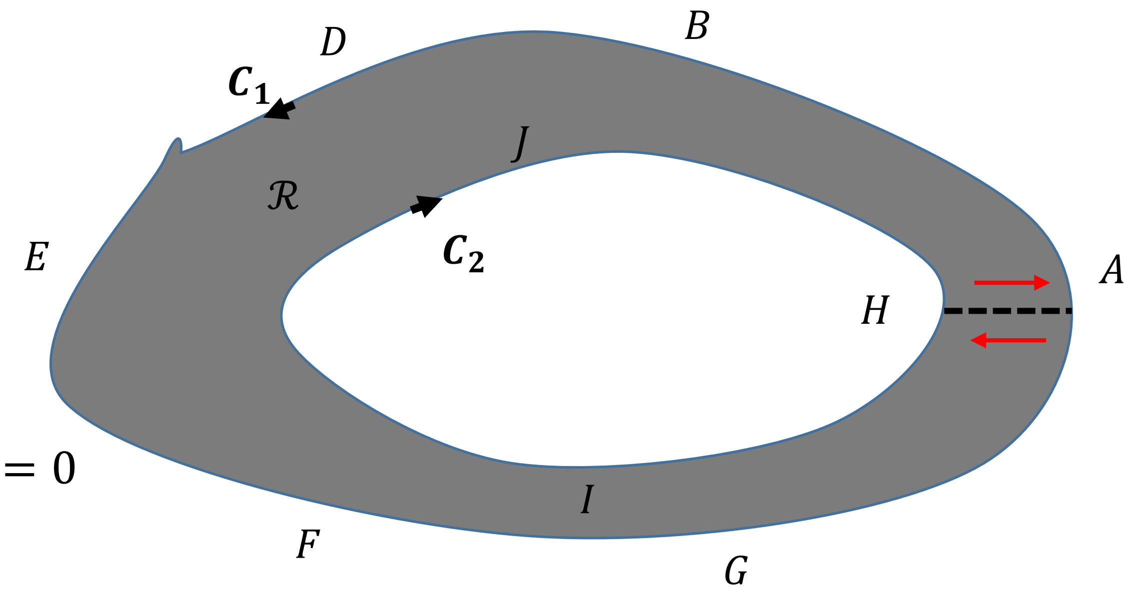
Q. Prove the Cauchy–Goursat theorem for multiply-connected regions.

Proof:

We shall present a proof for the multiply-connected region R bounded by the simple closed curves C_1 and C_2 as indicated in Fig.

Construct cross-cut **AH**. Then the region bounded by **ABDEFGAHIJHA** is simply-connected, So

$$\oint_{\mathbf{ABDEFGAHIJHA}} f(z)dz = 0$$



$$\Rightarrow \oint_{\mathbf{ABDEFGA}} f(z)dz + \oint_{\mathbf{AH}} f(z)dz + \oint_{\mathbf{HIJH}} f(z)dz + \oint_{\mathbf{HA}} f(z)dz = 0$$

$$\Rightarrow \oint_{\mathbf{ABDEFGA}} f(z)dz + \oint_{\mathbf{HIJH}} f(z)dz = 0$$

$$\Rightarrow \oint_{\mathbf{C}} f(z)dz = 0$$

where C is the complete boundary of R (consisting of ABDEFGA and HIJH) traversed in the sense that an observer walking on the boundary always has the region R on his/her left.

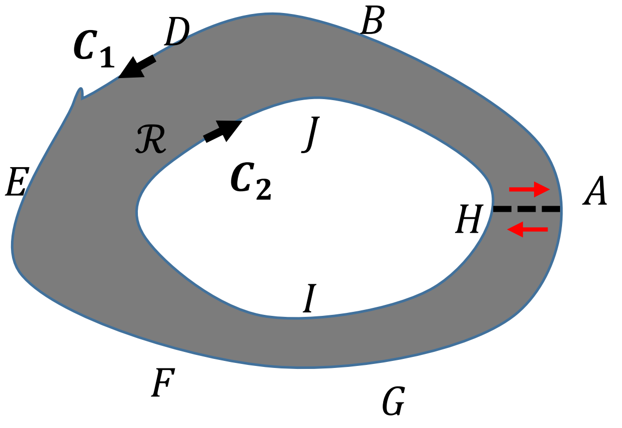
Cauchy's Theorem. The Cauchy–Goursat Theorem (page 129) (only focused on the results)

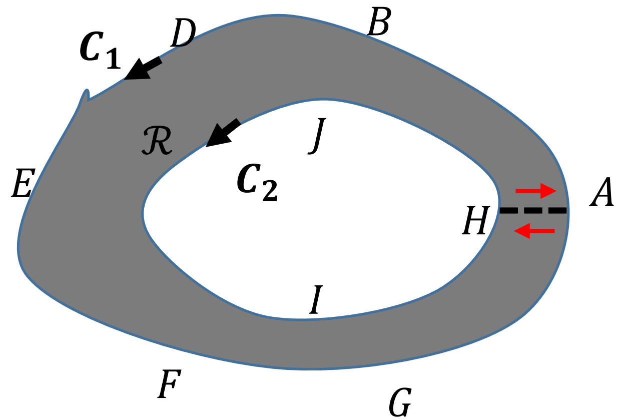
Note: Since,

$$\oint_{\mathbf{ABDEFGA}} f(z)dz + \oint_{\mathbf{HIJH}} f(z)dz = 0$$

$$\Rightarrow \oint_{\boldsymbol{C_1}} f(z)dz + \oint_{C_2} f(z)dz = 0$$

$$\Rightarrow \oint_{\boldsymbol{C_1}} f(z)dz = -\oint_{C_2} f(z)dz$$





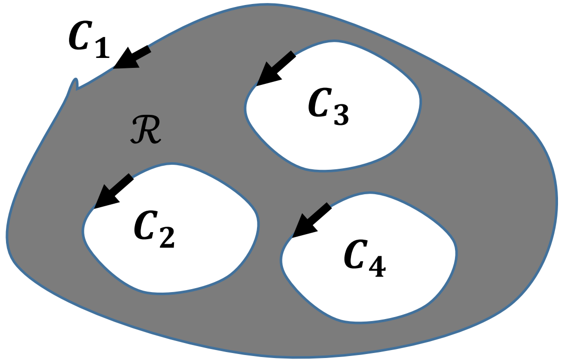
Now, if we take the integration on contour $\boldsymbol{C_1}$ as positive sense and on the contour $\boldsymbol{C_2}$ as negative sense as shown in figure, then

$$\oint_{\boldsymbol{C_1}} f(z)dz = \oint_{C_2} f(z)dz$$

Outer contour integration = Inner contour integration

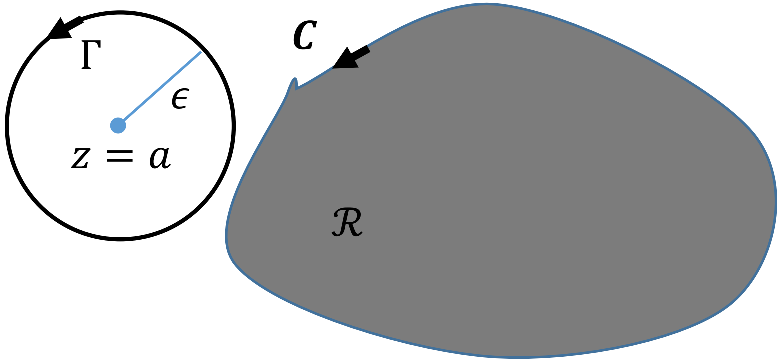
Similarly,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \oint_{C_4} f(z)dz$$



Q. Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z = a \text{ or } z = z_0$ is (a) outside C and (b) inside C.

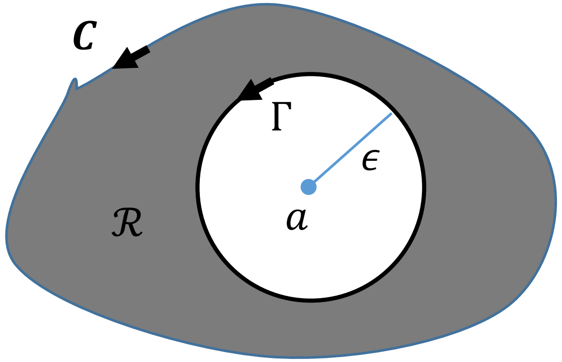
Solution: Here, the given function $f\ (z) = \frac{1}{z-a}$ has a singularity at $z = a$, so the function is blows up at this point and hence it is not analytic at $z = a$.



(a) If $z = a$ is outside C, then $f\ (z) = \frac{1}{z-a}$ is analytic everywhere inside and on C.

Hence, by Cauchy's theorem $\oint_C \frac{dz}{z-a} = 0$

(b) Suppose $z = a$ is inside C and let Γ be a circle of radius ϵ with center at $z = a$ so that Γ is inside C.



We can write

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \qquad \text{-----------(1)}$$

Now, on Γ

$$|z-a| = \epsilon$$

$$\Rightarrow |z-a| = \epsilon\left|e^{i\theta}\right|$$

$$\Rightarrow z-a = \epsilon e^{i\theta}$$

Equation of the circle

Since, $\left|e^{i\theta}\right|$=1

Here, $0 \le \theta < 2\pi$

Thus, since $dz = i\epsilon e^{i\theta} d\theta$, the right side of (1) becomes

$$\oint_C \frac{dz}{z-a} = \oint_0^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}}$$

$$= \oint_0^{2\pi} i d\theta$$

$$= i2\pi \text{ or } 2\pi\text{i}$$

This is the required contour integration value.

Cauchy's Integral Formula

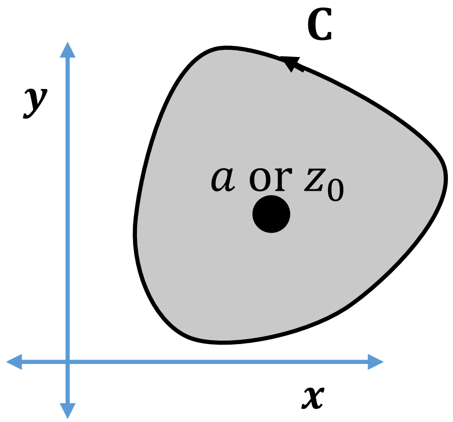
Statement (Cauchy's integral formula)

- Let f (z) be analytic inside and on a simple closed curve C and let a be any point inside C. Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}dz$$

where C is traversed in the positive (counterclockwise) sense.

When, n = 0 we get
(This is a special case)



Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Where, $n = 1, 2, 3, 4, 5, \ldots$

Q. What is the significant of this formula? Or Why it is so important?

Forward : If a function of a complex variable has a first derivative, i.e., is analytic, in a simply-connected region R, all its higher derivatives exist in R. This is not necessarily true for functions of real variables.

Backward : If a function $f(z)$ is analytic in a simple closed curve C, then we can find the integration of a function $G(z)$ such that $G(z) = \dfrac{f(z)}{(z-a)}$. The required integration will be

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = f^n(a) \times \frac{2\pi i}{n!}$$

Q. Evaluate $\oint_C \frac{1}{z-a} dz$ where C is any simple closed curve C and $z = a \; or \; z = z_0$ is ~~(a) outside C and~~ (b) inside C.

Solution: (b) We know that

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = f^n(a) \times \frac{2\pi i}{n!}$$

The given integration $\oint_C \frac{dz}{(z-a)^{0+1}}$.

Now by comparing with Cauchy's integral formula

$$f(z) = 1$$

$$n = 0$$

So, the required integration

$$\oint_C \frac{dz}{z-a} = 1 \times \frac{2\pi i}{0!}$$

$$= 2\pi i$$

When you become expert in this course

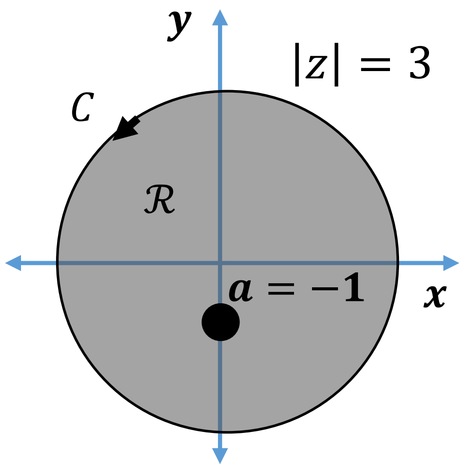
Using Cauchy integral formula

$$\oint_C \frac{f(z)}{(z-a)^{0+1}} dz = 1 \times \frac{2\pi i}{0!}$$

$$\Rightarrow \quad \oint_C \frac{f(z)}{(z-a)} dz = 2\pi i$$

This is the required integration.

Q. Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 3$.



Solution: Since, Cauchy's integral formula

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = f^n(a) \times \frac{2\pi i}{n!}$$

By comparing the given integration with Cauchy's integral formula

$$\oint_C \frac{e^{2z}}{(z+1)^{3+1}} dz = f^3(-1) \times \frac{2\pi i}{3!} \text{-----------} (i)$$

If the point '***a***' is outside the region then we can directly used Cauchy's theorem and will get the results as '**0**' i.e

$$\oint_C \frac{e^{2z}}{(z+1)^4} dz = 0$$

Now,

$$f(z) = e^{2z}$$

$$\Rightarrow \quad f^{/}(z) = 2 \times e^{2z}$$

$$\Rightarrow \quad f^{//}(z) = 4 \times e^{2z}$$

$$\Rightarrow \quad f^{///}(z) = 8 \times e^{2z}$$

$$\Rightarrow f^{///}(-1) = 8 \times e^{-2}$$

The required integration from equation (i)

$$\oint_C \frac{e^{2z}}{(z+1)^{3+1}} dz = 8 \times e^{-2} \times \frac{2\pi i}{3!}$$

$$= 8 \times e^{-2} \times \frac{2\pi i}{3 \times 2}$$

$$= \frac{8\,\pi i}{3\,e^2}$$

Q. Evaluate $\oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-1)(z-2)} dz$ where C is the circle **(a)** $|z| = 3$ and **(b)** $|z| = 2$.

Solution: Since,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

When, $z = 2$, we get

$$1 = B(2-1)$$

$$\Rightarrow B = 1$$

When, $z = 1$, we get

$$1 = A(1-2)$$

$$\Rightarrow A = -1$$

Now,

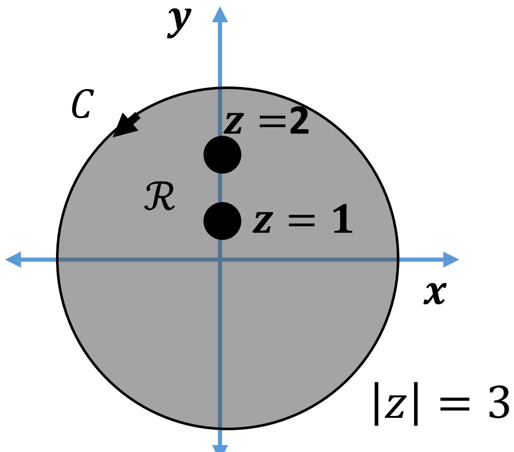
$$\frac{1}{(z-1)(z-2)} = \frac{(-1)}{(z-1)} + \frac{1}{(z-2)}$$

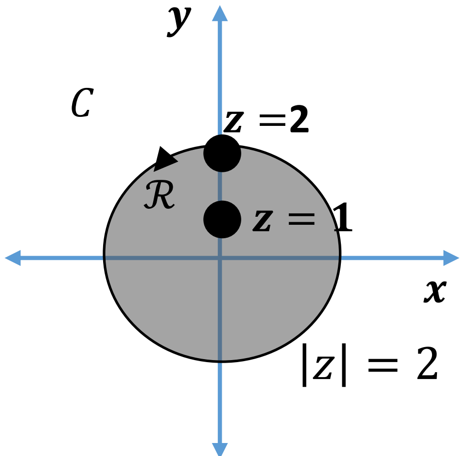
$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

So the given integration becomes,

$$\oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-1)(z-2)} dz = \oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-2)} dz - \oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-1)} dz$$

Since $z = 1$ and $z = 2$ are inside or on C and $(sin\pi z^2 + cos\pi z^2$ is analytic inside C.





By Cauchy's integral formula

$$\oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-1)(z-2)} dz = 2\pi i\ [(\sin(\pi 2))^2 + \cos(\pi 2)^2] - 2\pi i\ [(\sin(\pi))^2 + \cos(\pi)^2]$$

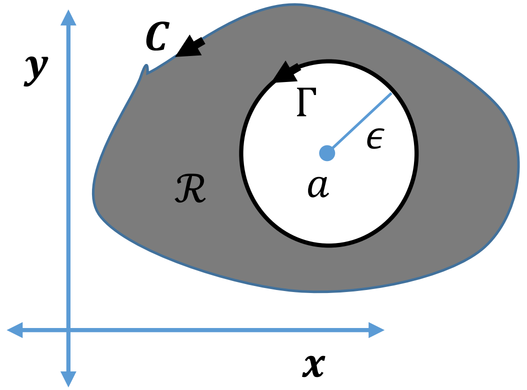
$$\oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-1)(z-2)} dz = 2\pi i\ - 2\pi i(-1)$$

$$\oint_C \frac{(sin\pi z^2 + cos\pi z^2)}{(z-1)(z-2)} dz = 4\pi i$$

This is the required integration.

Q. Let $f\,(z)$ be analytic inside and on the boundary C of a simply-connected region R. Prove Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}dz \qquad Or \qquad f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}}$$



where C is traversed in the positive (counterclockwise) sense.

Solution:

The function $\frac{f(z)}{(z-a)}$ is analytic inside and on C except at the point $z = a$. Now we can write,

$$\oint_C \frac{f(z)\,dz}{z-a} = \oint_\Gamma \frac{f(z)dz}{z-a} \qquad \text{-----------(1)}$$

Now, on Γ

$\lvert z-a\rvert = \epsilon$	Equation of the circle
$\Rightarrow \lvert z-a\rvert = \epsilon\lvert e^{i\theta}\rvert$	Since, $\lvert e^{i\theta}\rvert$=1
$\Rightarrow z-a = \epsilon e^{i\theta}$	Here, $0 \le \theta < 2\pi$

Thus, since $dz = i\epsilon e^{i\theta}d\theta$, the right hand side of equation (1) becomes

$$\oint_C \frac{f(z)dz}{z-a} = \oint_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta}{\epsilon e^{i\theta}}$$

$$= i\oint_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \qquad \text{-----------(2)}$$

Taking the limit of both sides of (2) and making use of the continuity of f (z), we have

$$\lim_{\epsilon\to 0}\oint_C \frac{f(z)dz}{z-a} = \lim_{\epsilon\to 0} i\oint_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

$$\oint_C \frac{f(z)dz}{z-a} = i\oint_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\,d\theta$$

$$\oint_C \frac{f(z)dz}{z-a} = i\oint_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\, d\theta$$

$$= i\oint_0^{2\pi} f(a)\, d\theta$$

$$= 2\pi i f(a)$$

so that we have

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a} \quad \text{------------(3)}$$

This is the required Cauchy's integral formula

Now, differentiating eq. (3) w.r.t ***a***, we will get

$$f^{/}(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^2}$$

Again, differentiating eq. (3) w.r.t **a** twice, we will get

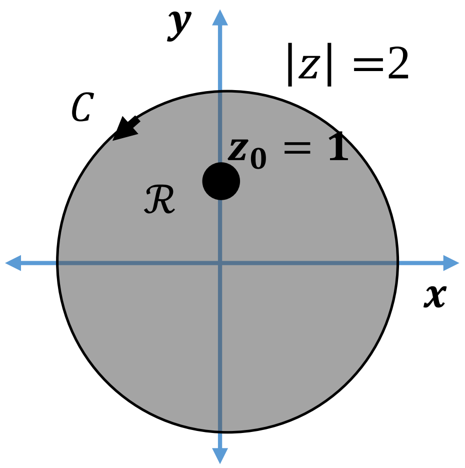
$$f^{//}(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^3}$$

Hence after differentiation eq. (3) ***n*** number of times we will get

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}}$$

This is the general form of Cauchy's integral formula.

Q. Evaluate $\oint_C \frac{e^z(z^2+1)}{(z-1)^2} dz$ where C is the circle **(a)** $|\boldsymbol{z}|$ **=2.**



Solution: Since, Cauchy's integral formula

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = f^n(a) \times \frac{2\pi i}{n!}$$

By comparing the given integration with Cauchy's integral formula

$$\oint_C \frac{e^z(z^2+1)}{(z-1)^{1+1}} dz = f^1(1) \times \frac{2\pi i}{1!} \quad ----------(i)$$

Now,

$$f(z) = e^z(z^2+1)$$

$$\Rightarrow \quad f^/(z) = e^z(2z) + e^z(z^2+1)$$

$$\Rightarrow \quad f^/(1) = e^1(2) + e^1(1^2+1)$$

$$= 4e$$

The required integration from equation (i)

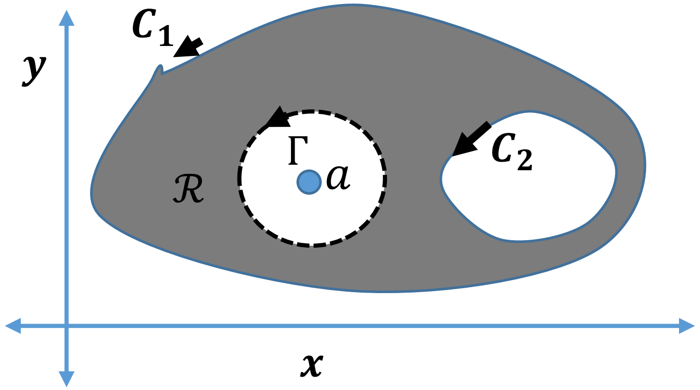
$$\oint_C \frac{e^z(z^2+1)}{(z-1)^{1+1}} dz = 4e \times \frac{2\pi i}{1!}$$

$$= 8\pi i e$$

If the integration is $\oint_C \frac{e^z(z^2+1)}{(z-3)^2} dz$ then the point **'*a*' *or* z_0** is outside the region. So, we can directly used Cauchy's theorem and will get the results as **'0' i.e**

$$\oint_C \frac{e^z(z^2+1)}{(z-1)^2} dz = 0$$

Q. Prove Cauchy's integral formula for multiply-connected regions.



Q. Evaluate $\oint_C \frac{(z-1)}{(z^2+1)}dz$ where C is the circle **(a)** $|\boldsymbol{z}-\boldsymbol{i}|$ **=1** and **(b)** $|\boldsymbol{z}|$ **=2**. (*2017, 2+2=4*)

Solution: Since,

$$\frac{z-1)}{(z^2+1)} = \frac{(z-1)}{(z+i)(z-i)}$$

$$= \frac{1}{2i}\left[\frac{(z-1)}{(z-i)} - \frac{(z-1)}{(z+i)}\right]$$

So the given integration becomes,

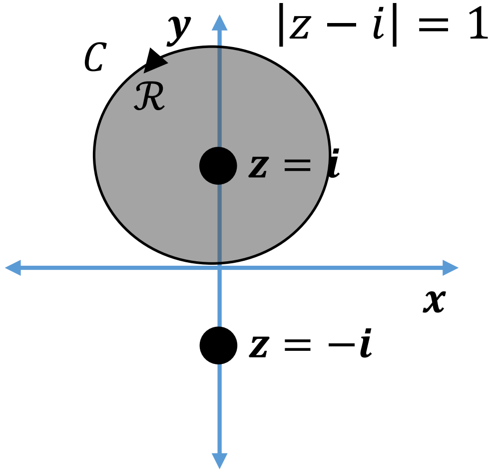
$$\oint_C \frac{z-1)}{(z^2+1)}dz = \frac{1}{2i}\left[\oint_C \frac{(z-1)}{(z-i)}dz - \oint_C \frac{(z-1)}{(z+i)}dz\right]$$

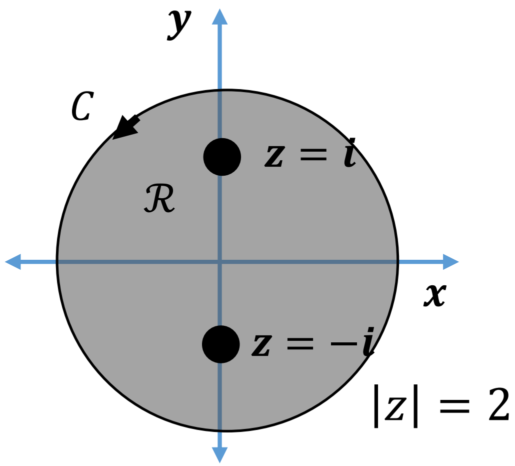
(a) When the region is $|z-i|$ =1, it is found that the singular point $z=-i$ is out the region, therefore

$$\oint_C \frac{z-1)}{(z^2+1)}dz = \frac{1}{2i}\left[\oint_C \frac{(z-1)}{(z-i)}dz - 0\right]$$

Here, $f(z) = z-1$

$$= \frac{1}{2i}[2\pi i \times f(i)] \qquad = \pi(i-1)$$





(b) When the region is $|z|$ =2, it is found that both the singular point $z=-i$ and $z=i$ is inside the region, therefore

$$\oint_C \frac{z-1)}{(z^2+1)}dz = \frac{1}{2i}\left[\oint_C \frac{(z-1)}{(z-i)}dz - -\oint_C \frac{(z-1)}{(z+i)}dz\right]$$

$$= \frac{1}{2i}[2\pi i(i-1) - 2\pi i(-i-1)]$$

$$= 2\pi i$$

Q. Evaluate $\oint_C \frac{1}{z} dz$ where C is the circle of unit radius. (*2017, 2*)

Solution: Using Cauchy integral formula

$$\oint_C \frac{1}{(z-0)^{0+1}} dz = 1 \times \frac{2\pi i}{0!}$$

$$\Rightarrow \quad \oint_C \frac{1}{z} dz = 2\pi i$$

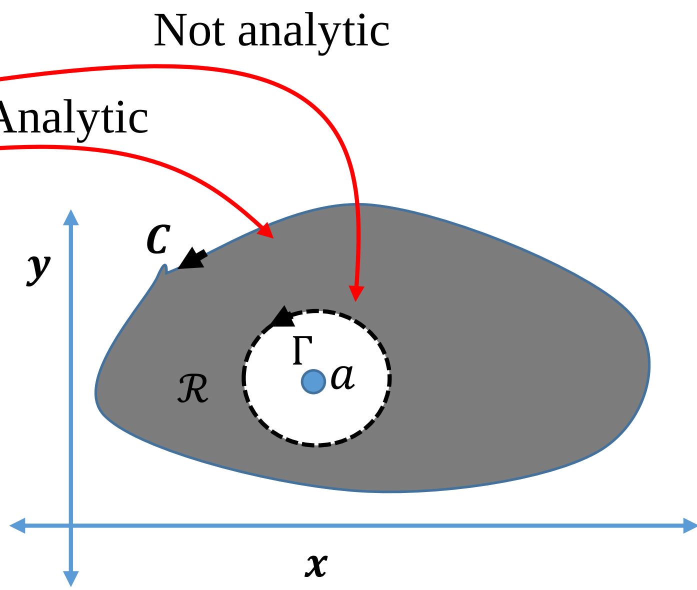
This is the required integration.

Residue theorem

Q. (a) Let F(z) be analytic inside and on a simple closed curve C except for a pole of order m at $z = a$ inside C. Prove that

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(b) How would you modify the result in (a) if more than one pole were inside C?



Solution: (a)

If $F(z)$ has a pole of order m at $z = a$, then $\boldsymbol{F(z) = f(z)/(z-a)^m}$ where $f(z)$ is analytic inside and on C, and $f(a) \neq 0$. Then, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^m}dz$$

$$= \frac{f^{(m-1)}(a)}{(m-1)!}$$

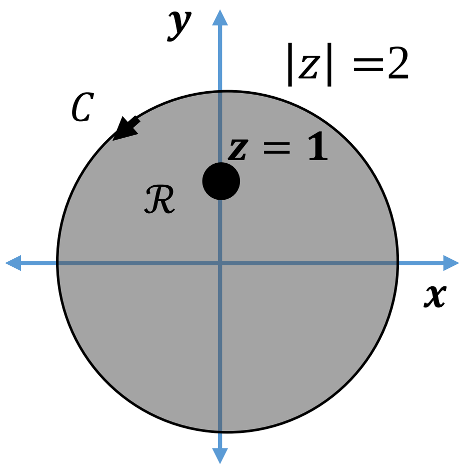
$$= \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{f(z)\}$$

$$= \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

$$= R$$

Here, R is called the **residues** of $F(z)$ at the poles $z = a$.

Q. Evaluate $\oint_C \frac{e^z(z^2+1)}{(z-1)^2}dz$ where C is the circle **(a)** $|z| = 2$.



Solution: Here, the pole $z = 1$ or $a = 1$ is inside the given region and the pole is order 2 i.e. $m = 2$.

Now, from Cauchy's integral formula

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)!}\,\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

$$\Rightarrow \oint_C F(z)\,dz = 2\pi i \times \lim_{z\to a}\frac{1}{(m-1)!}\,\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

$$\Rightarrow \oint_C \frac{e^z(z^2+1)}{(z-1)^2}dz = 2\pi i \times \lim_{z\to 1}\frac{1}{(2-1)!}\,\frac{d^{2-1}}{dz^{2-1}}\{(z-1)^2\frac{e^z(z^2+1)}{(z-1)^2}\}$$

$$= 2\pi i \times \lim_{z\to 1}\frac{d}{dz}\{e^z(z^2+1)\}$$

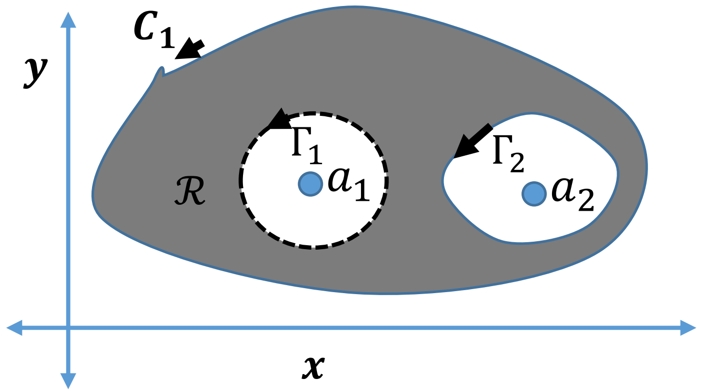
$$= 2\pi i \times \lim_{z\to 1}\{e^z(2z) + e^z(z^2+1)\}$$

$$= 2\pi i \times \{e^1(2) + e^1(1^2+1)\}$$

$$= 8\pi i e$$

(b) Suppose there are two poles at $z = a_1$ and $z = a_2$ inside C, of orders $m_1 and\ m_2$, respectively. Let Γ_1 and Γ_2 be circles inside C having radii ϵ_1 and ϵ_2 and centers at a_1 and a_2, respectively. Then

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} F(z)\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} F(z)\,dz \quad ---------(i)$$



If $F(z)$ has a pole of order m_1 at $z = a_1$, then

$$F(z) = \frac{f_1(z)}{(z-a_1)^{m_1}} \qquad \text{where } f_1(z) \text{ is analytic and } f_1(z) \neq 0$$

If $F(z)$ has a pole of order m_2 at $z = a_2$, then

$$F(z) = \frac{f_2(z)}{(z-a_2)^{m_2}} \qquad \text{where } f_2(z) \text{ is analytic and } f_2(z) \neq 0$$

So from equation (i)

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f_1(z)}{(z-a_1)^{m_1}}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f_2(z)}{(z-a_2)^{m_2}}\,dz$$

$$= \lim_{z\to a_1}\frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\{(z-a_1)^{m_1}F(z)\} + \lim_{z\to a_2}\frac{1}{(m_2-1)!}\frac{d^{m_2-1}}{dz^{m_2-1}}\{(z-a_1)^{m_2}F(z)\}$$

$$= R_1 + R_2$$

So we can write

$$\oint_C F(z)\,dz = 2\pi i(R_1 + R_2)$$

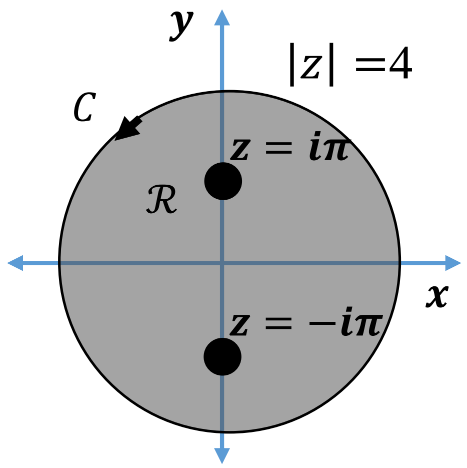
where $R_1\, and\ R_2$ are called the residues of $F(z)$ at the poles $z = a_1$ and $z = a_2$.

In general, if $F(z)$ has a number of poles inside C with residues R_1, R_2, . . ., then

$$\oint_C F(z)\,dz = 2\pi i(R_1 + R_2 + \cdots)$$
$$= 2\pi i\ (sum\ of\ the\ residues)$$

This result is called the ***residue theorem***.

Q. Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is the circle **(a) |*z*| =4.**



Solution: The poles of

$$\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+i\pi)^2(z-i\pi)^2}$$

are at $z = \pm i\pi$ inside C and are both of order two i.e. $m = 2$.

Now, residue at $z = i\pi$ is

$$\lim_{z\to i\pi} \frac{1}{1!}\,\frac{d}{dz}\left\{(z-i\pi)^2\frac{e^z}{(z+i\pi)^2(z-i\pi)^2}\right\} = \frac{\pi+i}{4\pi^3}$$

Similarly, residue at $z = -i\pi$ is

$$\lim_{z\to -i\pi} \frac{1}{1!}\,\frac{d}{dz}\left\{(z+i\pi)^2\frac{e^z}{(z+i\pi)^2(z-i\pi)^2}\right\} = \frac{\pi-i}{4\pi^3}$$

Therefore

$$\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\ (sum\ of\ residues) = 2\pi i\left(\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}$$

Q. Obtain the residue of the following

1. $f(z) = \frac{1}{z^2+a^2}$ where $a > 0$ — 2018, marks: 4
2. $f(z) = \frac{e^{iz}}{z^2+a^2}$ at $z = ia$ — 2017, marks: 2
3. $f(z) = \frac{e^z}{(z-i)^2}$ at $its\ pole$ — 2015, marks: 2
4. $f(z) = \frac{e^z}{(z-2)^3}$ at $its\ pole$ — 2013, marks: 2
4. $f(z) = \frac{z^2}{(1+z^2)^3}$ at $its\ pole$ — 2020, marks: 3

No need to submit. Just do it yourself as practice.

Q. For a function $f(z)$ which has a pole of order m at $z = z_0$, show that the residue of the function at that singular point is

$$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

2013, 2015, marks: 5

Note: Here, since marks is 5, so it is recommended that you should start from Cauchy's integral formula.

Taylor Series

Power Series:

- In mathematics, a **power series** (*in one variable*) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a)^1 + a_2(x-a)^2 + \cdots \qquad ---------(1)$$

where a_n represents the coefficient of the n^{th} term and a is a constant.

- In many situations a (the *center* of the series) is equal to zero, for instance when considering a **Maclaurin series**. In such cases, the power series takes the simpler form

$$\sum_{n=0}^{\infty} a_n(x)^n = a_0 + a_1x + a_2x^2 + \cdots$$

Taylor's theorem:

If $f\ (x)$ is differentiable in region, then $f(x)$ can be expand around a given point $\boldsymbol{a}$ as

$$f(x) = f(a) + f^{/}(a)(x-a) + \frac{f^{//}(a)}{2!}(x-a)^2 + \frac{f^{///}(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$= \sum_{n=0}^{\infty} a_n\,(x-a)^n$$

This is also called as power series.

Example: Any polynomial can be easily expressed as a power series around any center a.

For example $\boldsymbol{f(x) = x^2 + 2x + 3}$ can be written as a

(a) power series around the center $\boldsymbol{a = 0}$ as

$$f(x) = f(a) + f^{/}(a)(x-a) + \frac{f^{//}(a)}{2!}(x-a)^2 + \frac{f^{///}(a)}{3!}(x-a)^3 + \cdots$$

$$= f(0) + f^{/}(0)x + \frac{f^{//}(0)}{2!}x^2 + \frac{f^{///}(0)}{3!}x^3 + \cdots$$

$$= 3 + 2x + x^2 + 0 + 0 + \cdots$$

$$= \boldsymbol{3 + 2x + x^2}$$

$f(0) = 3$

$f^{1}(0) = (2x + 2\,)_{a=0} = 2$

$\frac{f^{//}(0)}{2!} = \frac{(2)_{a=0}}{2} = 1$

(b) power series around the center $\boldsymbol{a}$ =1 as

$$f(x) = f(1) + f^{/}(1)(x-1) + \frac{f^{//}(1)}{2!}(x-1)^2 + \cdots$$

$$= 6 + 4(x-1) + (x-1)^2 + 0 + 0$$

$$= 6 + 4x - 4 + x^2 - 2x + 1$$

$$= \boldsymbol{3 + 2x + x^2}$$

$f(1) = 6$

$f^{/}(1) = 4$

$\frac{f^{//}(1)}{2!} = 1$

4. Laurent and Taylors expansion (*only for concept*)

- Let represent the exponential function $f(x) = e^{x}$ by the infinite polynomial (power series).

- Since, here

$$f^{/}(x) = f^{//}(x) = f^{///}(x) = e^{x}$$

and

$$f^{/}(0) = f^{//}(0) = f^{///}(0) = e^{0} = 1$$

Now, the function can be represented as a power series using the Maclaurin's formula with $a_n = \frac{f^{n}(a)}{n!} = \frac{1}{n!}$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots + \frac{x^{n}}{n!} + \cdots$$

series

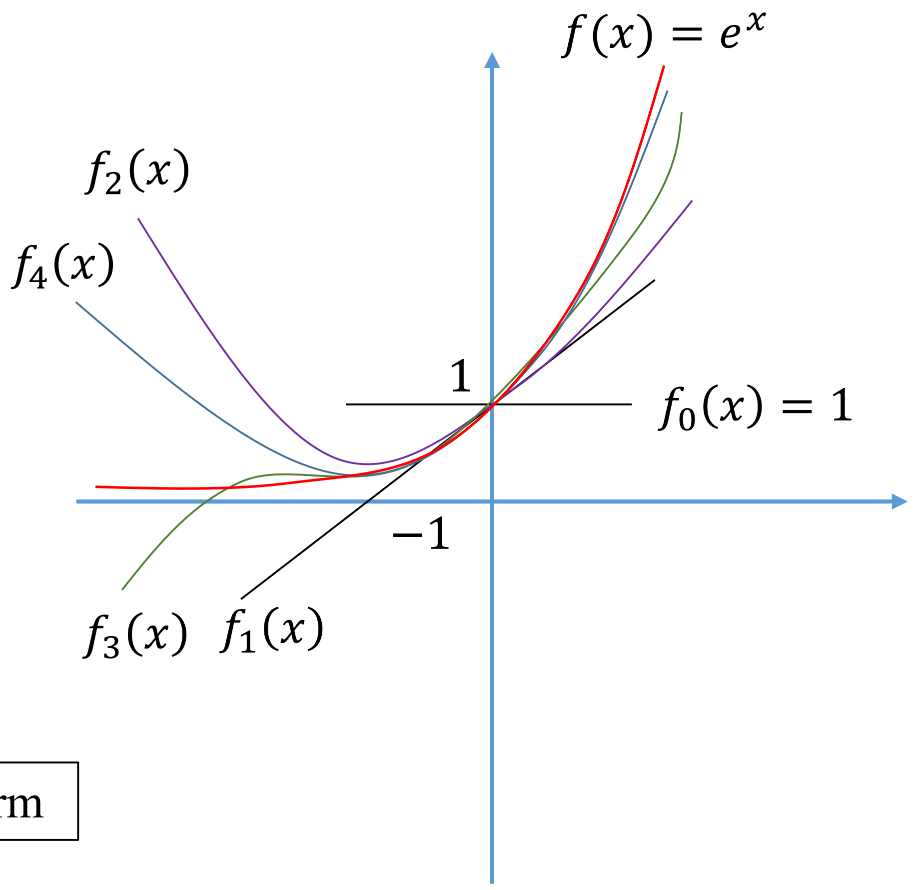
$f_0(x)$

$f_1(x)$

$f_2(x)$

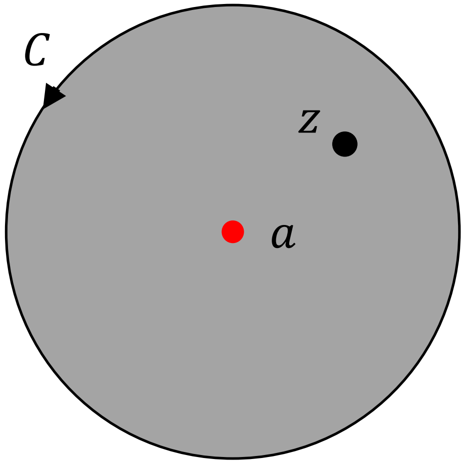
$f_3(x)$

Sequence's term



Taylor's theorem:

If $f\ (z)$ is analytic inside a circle C with center at a, then for all z inside C



$$f(z) = f(a) + f^{/}(a)(z-a) + \frac{f^{//}(a)}{2!}(z-a)^2 + \frac{f^{///}(a)}{3!}(z-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n\,(z-a)^n \qquad \textit{This is a power series}$$

This is called **Taylor's theorem** and the series is called a **Taylor series or expansion** for $f\ (z)$.

- The region of convergence of the series is given by $|z-a| < R$, where the radius of convergence R is the distance from $\boldsymbol{a}$ to the nearest singularity of the function $f\ (z)$. On $|z-a| = R$, the series may or may not converge. For $|z-a| > R$, the series diverges.
- If the nearest singularity of $f\ (z)$ is at infinity, the radius of convergence is infinite, i.e., the series converges for all z.
- If $a = 0$ in Taylor series, the resulting series is often called a **Maclaurin series**.

Some special series:

The following list shows some special series together with their regions of convergence

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad |z| < \infty$

2. $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad |z| < \infty$

3. $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad |z| < \infty$

4. $ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad |z| < 1$

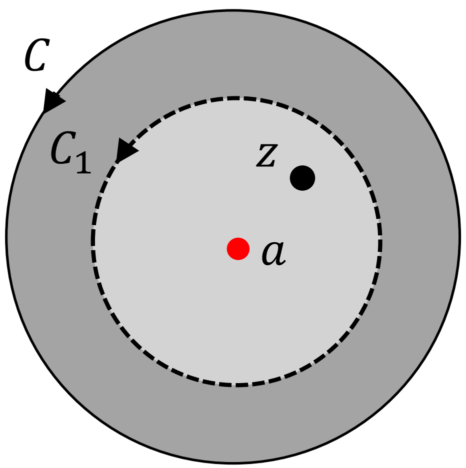
5. $(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots \qquad |z| < 1$

- $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$

Taylor's theorem Proof: Let z be any point inside C. Construct a circle C_1 with center at a and enclosing z. Then, by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z)}\,dw \qquad -------(1)$$

$$f^n(z) = \frac{n!}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z)^{n+1}}\,dw \qquad -------(2)$$



We have

$$\frac{1}{(w-z)} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\left\{\frac{1}{1-{(z-a)}/{(w-a)}}\right\}$$

$$= \frac{1}{(w-a)}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^{n}+\left(\frac{z-a}{w-a}\right)^{n+1}+\left(\frac{z-a}{w-a}\right)^{n+2}+\cdots\right\}$$

$$= \frac{1}{(w-a)}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^{n}\left[1+\left(\frac{z-a}{w-a}\right)^{1}+\left(\frac{z-a}{w-a}\right)^{2}+\cdots\right]\right\}$$

$$= \frac{1}{(w-a)}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^{n}\left[\frac{1}{1-{(z-a)}/{(w-a)}}\right]\right\}$$

$$= \frac{1}{(w-a)}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^{n}\frac{(w-a)}{(w-z)}\right\}$$

$$= \frac{1}{(w-a)} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{(w-z)} \qquad -------(3)$$

Now, multiplying both side of equation (3) by $f(w)/2\pi i$ and taking contour integration, thereafter using equation (1) we get

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)} dw + \frac{(z-a)}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2} dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^n} dw + U_n \qquad -------(4)$$

Where

$$U_n = \frac{1}{2\pi i}\oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{(w-z)} dw$$

Now, using equation (2), equation (4) becomes

$$f(z) = f(a) + f^{/}(a)(z-a) + \frac{f^{//}(a)}{2!}(z-a)^2 + \frac{f^{///}(a)}{3!}(z-a)^3 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + U_n \qquad -------(5)$$

If we can now show that $\lim_{n\to\infty} U_n = 0$, we will have proved the required result. To do this, we note that since w is on C_1,

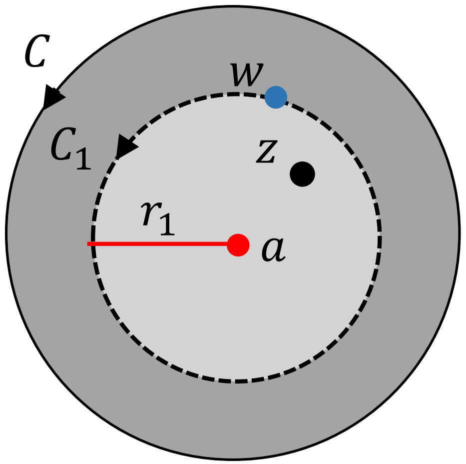
$$\left|\frac{z-a}{w-a}\right| = \gamma < 1$$

Where γ is a constant.

Also, we have $|f(w)| < M$, where M is a constant, and

$$|w - z| = |(w-a) - (z-a)| \geq r_1 - |z-a|$$

where r_1 is the radius of C_1. Now taking modulus of U_n we have



$$|U_n| = \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{(w-z)}dw\right|$$

$$\leq \frac{1}{2\pi}\left|\frac{z-a}{w-a}\right|^n \frac{|f(w)|}{|(w-z)|}\left|\oint_{C_1} dw\right|$$

$$= \frac{1}{2\pi}\ \frac{\gamma^n M}{r_1 - |z-a|}\ 2\pi r_1$$

$$= \frac{\gamma^n M r_1}{r_1 - |z-a|}$$

$$\left|\oint_C f(z)dz\right| \leq ML$$

where $|f(z)| \leq M$, i.e., M is an upper bound of $|f(z)|$ on C, and L is the length of C.

Or simply

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$|2 + (-3)| \leq |2| + |-3|$$

$$|-1| \leq 2 + 3$$

$$1 \leq 5$$

Now, taking the limit, $\lim_{n\to\infty}|U_n| = \lim_{n\to\infty}\frac{\gamma^n M r_1}{r_1 - |z-a|} = 0$

So from equation (5)

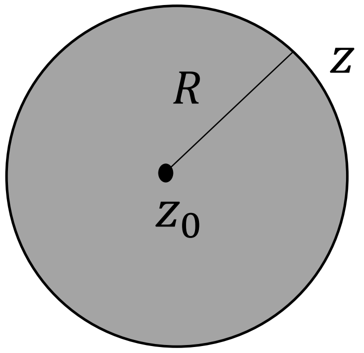
$$f(z) = f(a) + f^{/}(a)(x-a) + \frac{f^{//}(a)}{2!}(x-a)^2 + \frac{f^{///}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

Hence proved

Power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} Z_n$$

Circle of convergence



The power series is convergence when the reference point (z_0) about which we do the expansion is such that

$$|z - z_0| < R$$

Here, the R is called **radius of convergence**.

Now, using using ratio test,

$$L = \lim_{n\to\infty}\left|\frac{Z_{n+1}}{Z_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||z - z_0|$$

Ratio test

Let $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = L$. Then $\sum U_n$ converges if L<1 and diverges if L>1. If L=1, the test fails.

So according to ratio test the power series will be convergence when

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||z - z_0| < 1$$

$$= l \times R < 1$$

$$\therefore \textit{Radius of convergence}\ \mathbf{R} = \frac{\mathbf{1}}{\boldsymbol{l}}$$

Q1. Let $f(z) = \ln(1+z)$, then
(a) Expand $f(z)$ in a Taylor series about $z = 0$.
(b) Determine the region of convergence for the series in (a).
(c) Expand $\ln(1+z)/(1-z)$ in a Taylor series about $z = 0$.

Solution: We know the Taylor expansion

$$f(z) = f(a) + f^{/}(a)(z-a) + \frac{f^{//}(a)}{2!}(z-a)^2 + \frac{f^{///}(a)}{3!}(z-a)^3 + \cdots$$

(a) We need to expand $f(z)$ in a Taylor series about $z = 0$. So,

$$f(z) = f(0) + f^{/}(0)\, z + \frac{f^{//}(0)}{2!} z^2 + \frac{f^{///}(0)}{3!} z^3 + \cdots \quad -------(1)$$

Now,

$f(z) = \ln(1+z)$; $\quad f(0) = 0$

$f^{/}(z) = 1/(1+z)$; $\quad f^{/}(0) = 1$

$f^{//}(z) = -1/(1+z)^2$; $\quad f^{//}(0) = -1$

$f^{///}(z) = (-1)(-2)/(1+z)^3$; $\quad f^{///}(0) = 2!$

⋮

$f^{(n+1)}(z) = (-1)^n\, n!/(1+z)^{(n+1)}$; $\quad f^{(n+1)}(0) = (-1)^n$

Now, from equation (1)

$$f(z) = f(0) + f^{/}(0)\, z + \frac{f^{//}(0)}{2!} z^2 + \frac{f^{///}(0)}{3!} z^3 + \cdots$$

$$= 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$f(z) = \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

(b) The n^{th} term of the Taylor expansion is $U_n = (-1)^{n-1} z^n / n$. Using using ratio test,

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = \lim_{n\to\infty}\left|\frac{z}{\frac{n+1}{n}}\right| = \lim_{n\to\infty}\left|\frac{z}{1+\frac{1}{n}}\right| = |z|$$

and the series converges for $|z| < 1$.

Ratio test

Let $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = L$. Then $\sum U_n$ converges (absolutely) if L<1 and diverges if L>1. If L=1, the test fails.

(c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

both series convergent for $|z| < 1$. By subtraction, we have

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots = \sum_{n=0}^{\infty} \frac{2\, z^{2n+1}}{2n+1}\right.$$

Which converges for $|z| < 1$

Q1. (a) Expand $f\ (z) = \sin z$ in a Taylor series about $z = \pi/4$.

Solution: We know the Taylor expansion

$$f(z) = f(a) + f^{/}(a)(z-a) + \frac{f^{//}(a)}{2!}(z-a)^2 + \frac{f^{///}(a)}{3!}(z-a)^3 + \cdots$$

We need to expand $f\ (z)$ in a Taylor series about $z = \pi/4$. So,

$$f(z) = f(\pi/4) + f^{/}(\pi/4)(z-\pi/4) + \frac{f^{//}(\pi/4)}{2!}(z-\pi/4)^2 + \frac{f^{///}(\pi/4)}{3!}(z-\pi/4)^3 + \cdots \qquad ------ -(1)$$

Now,

$f(z) = \sin z$; $\quad f(\pi/4) = \sqrt{2}/2$

$f^{/}(z) = \cos z$; $\quad f(\pi/4) = \sqrt{2}/2$

$f^{//}(z) = -\sin z$; $\quad f(\pi/4) = -\sqrt{2}/2$

$f^{///}(z) = -\cos z$; $\quad f(\pi/4) = -\sqrt{2}/2$

Now, from equation (1)

$$\sin z = f(\pi/4) + f^{/}(\pi/4)(z-\pi/4) + \frac{f^{//}(\pi/4)}{2!}(z-\pi/4)^2 + \cdots$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(z-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\ .2!}\left(z-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\ .3!}(z-\frac{\pi}{4})^3 + \cdots$$

$$= \frac{\sqrt{2}}{2}(1 + \left(z-\frac{\pi}{4}\right) - \frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{3!}(z-\frac{\pi}{4})^3 + \cdots$$

Q1. (a) Expand $f\ (z) = \frac{1}{1-z}$ in a Taylor series about $z = i$. Also find the radius of convergence.

Solution: We know the Taylor expansion

$$f(z) = f(a) + f^{/}(a)(z-a) + \frac{f^{//}(a)}{2!}(z-a)^2 + \frac{f^{///}(a)}{3!}(z-a)^3 + \cdots$$

We need to expand $f\ (z)$ in a Taylor series about $z = i$. So,

$$f(z) = f(i) + f^{/}(i)(z-i) + \frac{f^{//}(i)}{2!}(z-i)^2 + \frac{f^{///}(i)}{3!}(z-i)^3 + \cdots \quad ------- -(1)$$

Now,

$f(z) = 1/1-z;$ $\quad f(i) = 1/1-i$

$f^{/}(z) = 1/(1-z)^2;$ $\quad f^{/}(i) = 1/(1-i)^2$

$f^{//}(z) = 2/(1-z)^3;$ $\quad f^{//}(i) = 2/(1-i)^3$

$f^{///}(z) = 3!/(1-z)^4;$ $\quad f^{///}(i) = 3!/(1-i)^4$

Now, from equation (1)

$$f\ (z) = \frac{1}{1-i}\left[1 + \frac{z-i}{1-i} + \frac{(z-i)^2}{(1-i)^2} + \cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

$$= \sum_{n=0}^{\infty} a_n (z-i)^n$$

$$a_n = \frac{1}{(1-i)^{n+1}}$$

The required Taylor's series is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-i)^n$$

$$a_n = \frac{1}{(1-i)^{n+1}}$$

Ratio test

Let $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = L$. Then $\sum U_n$ converges (absolutely) if L<1 and diverges if L>1. If L=1, the test fails.

Now, using using ratio test,

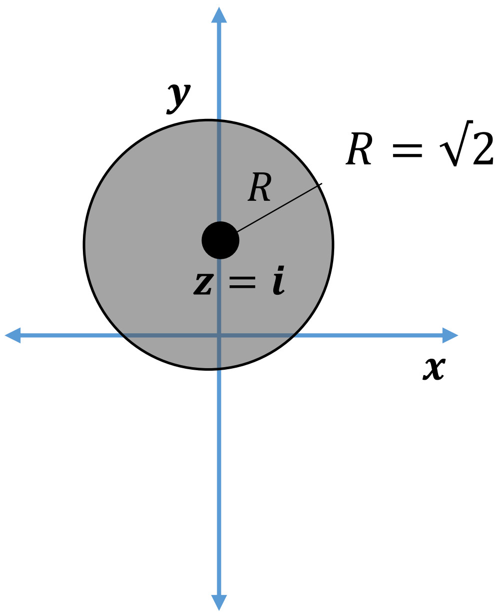
$$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{1-i}\right| = \lim_{n\to\infty} \left|\frac{1+i}{(1+i)(1-i)}\right|$$

$$= \left|\frac{1+i}{2}\right|$$

$$= \frac{1}{\sqrt{2}}$$

So, the radius of convergence.

$$\mathrm{R} = \frac{1}{l} = \sqrt{2}$$



Q. Find the first three terms of Taylor expansion of $f(z) = {}^{1}/_{z^2+4}$ about $z = -i$ and give the region of convergence. 2021

Laurent Series

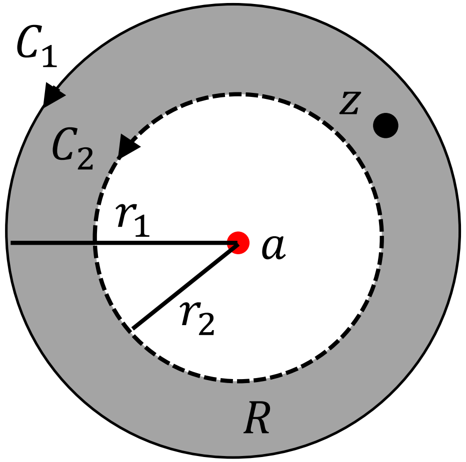
Laurent's theorem:

Suppose $f\ (z)$ is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles C_1 and C_2 with center at a and respective radii r_1 and r_2 $(r1 > r2)$ (see Fig. 6-5). Then for all z in R,

Principal part + Analytic part

$$f(z) = \sum_{n=1}^{\infty} a_{-n}(z - z_0\ )^{-n} + \sum_{n=0}^{\infty} a_n(z - z_0\ )^n$$

$$= \cdots + \frac{a_{-n}}{\left(z - z_0\ \right)^n} + \frac{a_{-n+1}}{\left(z - z_0\ \right)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_n\left(z - z_0\ \right)^1 + \cdots$$



Where

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}dw \qquad n = 0,1,2,3 \ldots..$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{-n+1}}dw \qquad n = 1,2,3 \ldots..$$

Laurent's theorem proof: (Assignment) (Marks:15)

Q. Find Laurent series about the indicated singularity for each of the following functions

1. $f(z) = \frac{sinz}{z}$; $\quad z = 0$

1. solution.:

$$f(z) = \frac{sinz}{z}$$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

= No principal part

So, this is removable singularity

2. $f(z) = \frac{sinz}{z^2}$; $\quad z = 0$

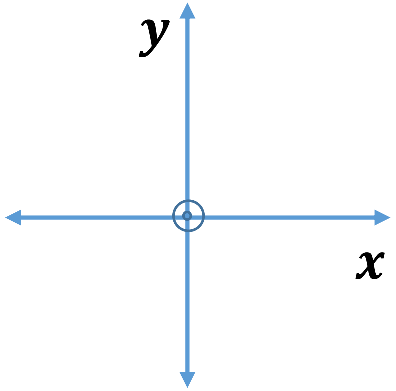
2. solution.:

$$f(z) = \frac{sinz}{z^2}$$

$$= \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots$$

So, isolated singularity of order 1.



3. $f(z) = \frac{e^{2z}}{(z-1)^3}$; $\quad z = 1$

3. solution.:

Let $z - 1 = u$. Then $z = u + 1$ and

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

$$= \frac{e^{2(u+1)}}{(u)^3}$$

$$= \frac{e^2}{u^3}. e^{2u}$$

$$= \frac{e^2}{u^3}\left[1 + 2u + \frac{(2u)^2}{2!} + \cdots\right]$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

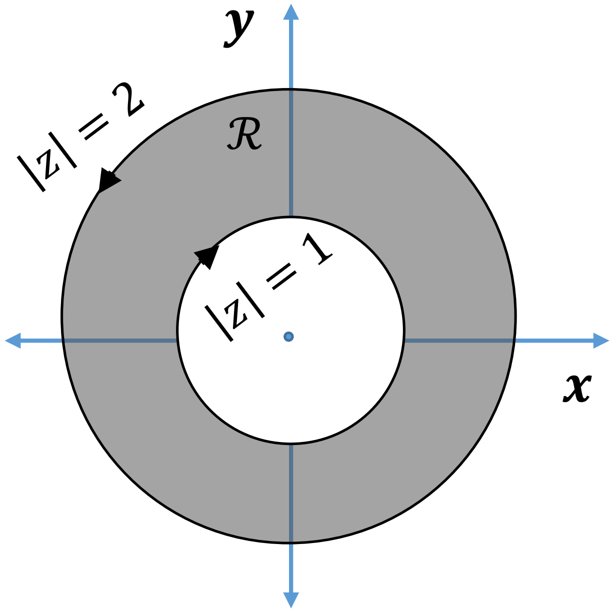
$z = 1$ is a pole of order 3, or triple pole. The series converges for all values of $z \neq 1$.

Q. Expand $f\,(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent series valid for $1 < |z| < 2$

Ans.: The given function

$$f\,(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

- $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ $|z| < 1$



Here, the given region

$$|z| < 2 \text{ so } \frac{|z|}{2} < 1$$

and

$$|z| > 1 \text{ so } \frac{1}{|z|} < 1$$

We have to remember these two condition while expanding the function

Now,

$$f\,(z) = \frac{1}{(-2)[1 - \frac{z}{2}]} - \frac{1}{z(1 - \frac{1}{z})}$$

$$f\,(z) = \left(-\frac{1}{2}\right)\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= \ldots - \left(\frac{1}{z}\right)^3 - \left(\frac{1}{z}\right)^2 - \frac{1}{z} - \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{z^3}$$

$$= \sum_{n=1}^{\infty} a_{-n}(z - z_0\,)^{-n} + \sum_{n=0}^{\infty} a_n(z - z_0\,)^n$$

Consequences of Cauchy Residue Theorem

Evaluation of Definite Integrals

Evaluation of definite integrals

- The evaluation of ***definite integrals*** is often achieved by using the ***residue theorem*** together with a suitable ***function f*** (z) and a suitable ***closed path or contour C***, the choice of which may require great ingenuity.

The following types are most common in practice:

1. $\int_0^{2\pi} F(sin\theta, cos\theta)d\theta$; where F$(sin\theta, cos\theta)$ is a rational function of $sin\theta$ and $cos\theta$.

2. $\int_{-\infty}^{+\infty} F(x)dx$; where F(x) is a rational function.

3. $\int_{-\infty}^{+\infty} F(x)\{ \begin{matrix} \cos mx \\ \sin mx \end{matrix} \}dx$; where F(x) is a rational function.

Convert the rational function into a suitable complex function i.e. F(z) → Choose a suitable contour C to apply CRT → Use CRT to solve the problem

Evaluation of definite integrals: $\int_0^{2\pi} F(sin\theta, cos\theta)d\theta$

$Q. Evaluate \int_0^{2\pi} \frac{d\theta}{5+4sin\theta}$

Solution:

Let $z = e^{i\theta}$. Then we know

$$sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \quad = \frac{z-z^{-1}}{2i}$$

$$cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad = \frac{z+z^{-1}}{2}$$

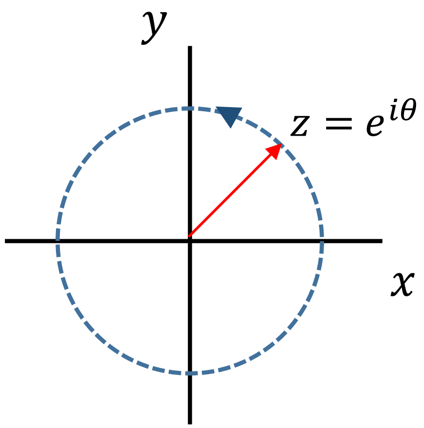
And

$$z = e^{i\theta}$$

$$\therefore \frac{dz}{d\theta} = ie^{i\theta}$$

$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}}$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$



so that

$$\int_0^{2\pi} \frac{d\theta}{5+4sin\theta} = \oint_C \frac{\frac{dz}{iz}}{5+4.(z-z^{-1})/2i}$$

$$= \oint_C \frac{dz}{2z^2+5iz-2} \qquad = \oint_C f(z)dz \qquad \text{-----------------(1)}$$

where C is the circle of unit radius with center at the origin as shown in fig.

Now the poles of $f(z)dz$

$$z = \frac{-5i \pm \sqrt{(5i)^2 - 4\times 2\times(-2)}}{2\times 2}$$

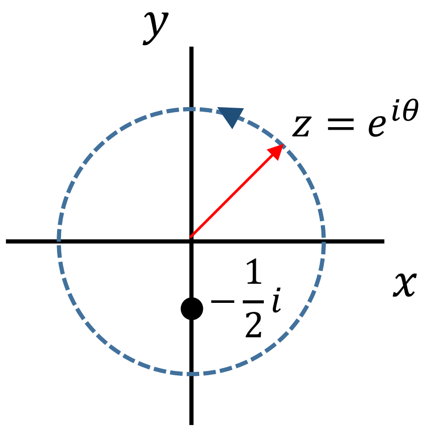
$$z = \frac{-5i \pm 3i}{4}$$

$$z = -\frac{1}{2}i, -2i$$

Therefore we get two poles

$$z_1 = -\frac{1}{2}i \qquad z_2 = -2i$$

But out of these two, only z_1 lies inside C



Now residue of $f(z)$ at $z_1 = -\frac{1}{2}i$

$$Res\left[f(z), (-\frac{1}{2}i)\right] = \lim_{z\to z_1} \frac{1}{(m-1)!}\frac{d^{m-1}}{d^{m-1}}\{(z-a)^m f(z)\}$$

$$= \lim_{z\to -\frac{1}{2}i} \frac{1}{(1-1)!}\frac{d^{1-1}}{d^{1-1}}\left\{\left(z+\frac{1}{2}i\right)^1 . \frac{1}{2z^2+5iz-2}\right\}$$

$$= \lim_{z\to -\frac{1}{2}i} \left(z+\frac{1}{2}i\right) . \frac{1}{2\times\left(z+\frac{1}{2}i\right)(z+2i)}$$

$$= \lim_{z\to -\frac{1}{2}i} \frac{1}{2(z+2i)}$$

$$= \frac{1}{2(-\frac{i}{2}+2i)}$$

$$Res\left[f(z), (-\frac{1}{2}i)\right] = \frac{1}{3i}$$

Now, apply Cauchy residue theorem, from equation (1)

$$\int_0^{2\pi} \frac{d\theta}{5+4sin\theta} = \oint_C \frac{dz}{2z^2+5iz-2}$$

$$= 2\pi i \times Res\left[f(z), (-\frac{1}{2}i)\right]$$

$$= 2\pi i \times \frac{1}{3i}$$

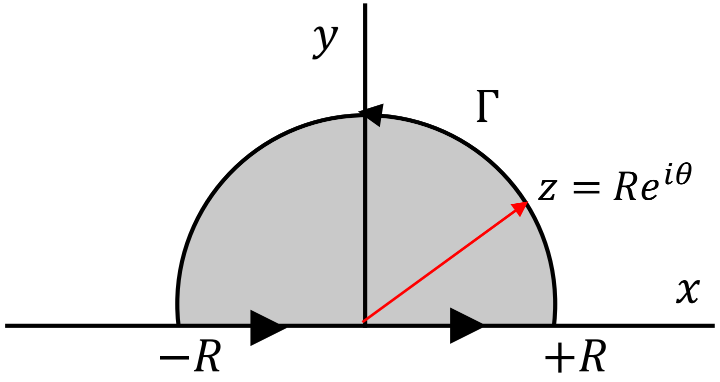
$$= \frac{2}{3}\pi$$

This is the required integration.

Evaluation of definite integrals: $\int_{-\infty}^{+\infty} F(x)dx$

❖ Suppose the given integral is $\int_{-\infty}^{+\infty} F(x)dx$

- Consider $\oint_C f(z)dz$ along a contour C consisting of the line along the x axis from $-R$ to $+R$ and the semicircle Γ above the x axis having this line as diameter.



Step 1: $F(x) \Rightarrow F(z)$

Step 2: Choose the contour i.e.

$$\oint_C F(z)dz = \int_\Gamma F(z)dz + \int_{-R}^{+R} F(x)dx$$

$$2\pi i \times \sum_{k=1}^{n} Res[F(z), z_k] = \int_\Gamma F(z)dz + \int_{-R}^{+R} F(x)dx$$

Step 3: We will take limit $R \to \infty$. After taking limit we will found that

$$\lim_{R\to\infty} \int_\Gamma F(z)dz = 0$$

This implies that

$$\lim_{R\to\infty} \int_{-R}^{+R} F(x)dx = 2\pi i \times \sum Res[F(z), z_k]$$

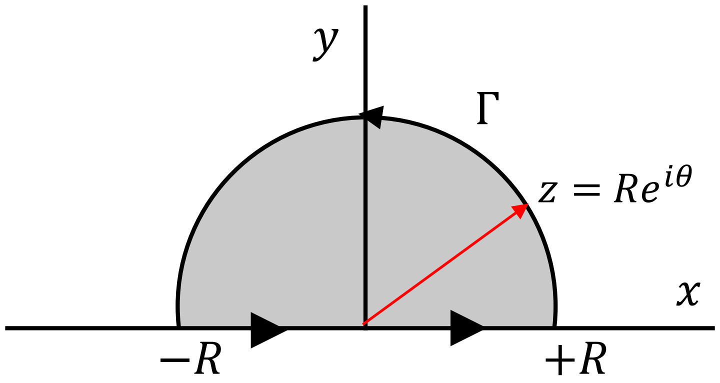
$$\Rightarrow \int_{-\infty}^{+\infty} F(x)dx = 2\pi i \times \sum Res[F(z), z_k]$$

This is the required integration

$Q. Evaluate \int_{-\infty}^{+\infty} \frac{dx}{(x^6+1)}$

Solution:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^6+1)} = \oint_C \frac{dz}{(z^6+1)}$$



where the contour C consisting of the line along the x axis from $-R$ to $+R$ and the semicircle Γ above the x axis having this line as diameter.

Now poles of $f(z) = 1/(z^6+1)$ are (i.e. solution of $z^6 + 1$=0)

$$e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}\ and\ e^{11\pi i/6}$$

Out of these 6 poles only $e^{\pi i/6}, e^{3\pi i/6}\ and\ e^{5\pi i/6}$ are lies inside the contour.

Now residues at poles $e^{\pi i/6}, e^{3\pi i/6}$ *and* $e^{5\pi i/6}$

$$Res[f(z),(e^{\pi i/6})] = \lim_{z\to e^{\pi i/6}} \left\{(z - e^{\pi i/6}).\frac{1}{z^6+1}\right\}$$

$$= \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6} e^{-5\pi i/6}$$

Similarly

$$Res[f(z),(e^{3\pi i/6})] = \lim_{z\to e^{3\pi i/6}} \left\{(z - e^{3\pi i/6}).\frac{1}{z^6+1}\right\} = \frac{1}{6} e^{-5\pi i/2}$$

and

$$Res[f(z),(e^{5\pi i/6})] = \lim_{z\to e^{5\pi i/6}} \left\{(z - e^{5\pi i/6}).\frac{1}{z^6+1}\right\} = \frac{1}{6} e^{-25\pi i/6}$$

$$z^6 = (e^{\pi i/6})^6 = e^{i\pi} = \cos(i\pi) + i\sin(i\pi) = -1$$

$L'Hospital's\ rule$

If $\lim_{z\to c} \frac{f(z)}{g(z)}$ undetermined i.e.

$$\lim_{z\to c} \frac{f(z)}{g(z)} = \frac{0}{0} \quad or \quad \lim_{z\to c} \frac{f(z)}{g(z)} = \frac{\infty}{\infty}$$

Then $\lim_{z\to c} \frac{f(z)}{g(z)}$ can be written as

$$\lim_{z\to c} \frac{f(z)}{g(z)} = \lim_{z\to c} \frac{f'(z)}{g'(z)}$$

Evaluation of definite integrals: $\int_{-\infty}^{+\infty} F(x)dx$

Thus, from Cauchy residue theorem

$$\oint_C \frac{dz}{(z^6+1)} = 2\pi i\left\{\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6}e^{-5\pi i/2} + \frac{1}{6}e^{-25\pi i/6}\right\}$$

$$\oint_C \frac{dz}{(z^6+1)} = 2\pi i\{-cos30 - isin30 + 0 - i + cos30 - isin30\}$$

$$\Rightarrow \int_\Gamma \frac{dz}{(z^6+1)} + \int_{-R}^{+R} \frac{dx}{(x^6+1)} = \frac{2\pi}{3} \qquad \text{------------------(1)}$$

$$\Rightarrow I_1 + I_2 = \frac{2\pi}{3}$$

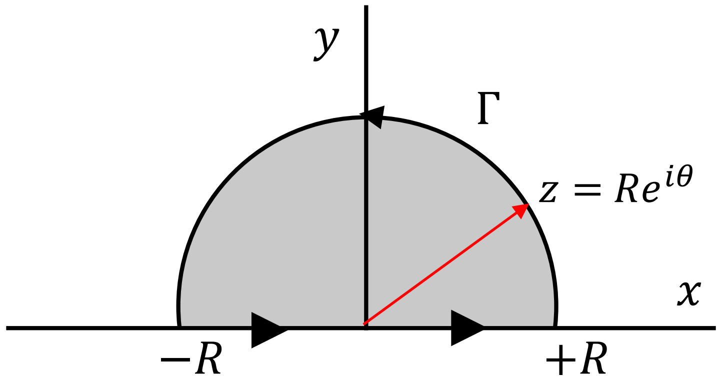
Now

$$I_1 = \int_\Gamma \frac{dz}{(z^6+1)} = \int_\Gamma \frac{iRe^{i\theta}d\theta}{R^6e^{i6\theta}+1}$$

$$z = Re^{i\theta}$$

$$\therefore \frac{dz}{d\theta} = iRe^{i\theta}$$

$$\Rightarrow dz = iRe^{i\theta}d\theta$$



or $|I_1| \le \int_\Gamma \frac{|iRe^{i\theta}d\theta|}{|R^6e^{i6\theta}+1|} \rightarrow convergent$

$$e.g. |5-3-1| \le \{|5| + |-3| + |-1|\}$$

Therefore when the limit $R \rightarrow \infty$, then I_1 becomes

$$\lim_{R\to\infty}|I_1| = \lim_{R\to\infty}\int_\Gamma \frac{|iRe^{i\theta}d\theta|}{|R^6e^{i6\theta}+1|} = 0$$

Evaluation of definite integrals: $\int_{-\infty}^{+\infty} F(x)dx$

Now taking limit on both side of equation (1)

$$\lim_{R\to\infty}\int_{\Gamma}\frac{dz}{(z^6+1)}+\lim_{R\to\infty}\int_{-R}^{+R}\frac{dx}{(x^6+1)}=\frac{2\pi}{3}$$

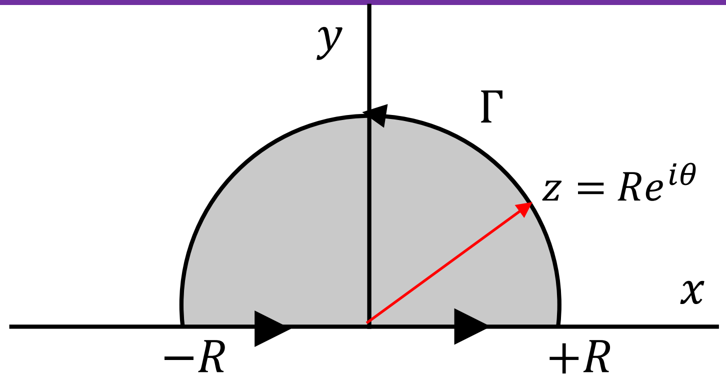
$$\Rightarrow 0+\lim_{R\to\infty}\int_{-R}^{+R}\frac{dx}{(x^6+1)}=\frac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{+\infty}\frac{dx}{(x^6+1)}=\frac{2\pi}{3}$$

This is the required integration.

$$Also, \int_{-\infty}^{+\infty}\frac{dx}{(x^6+1)}=\frac{\pi}{3}$$

$Q.\,Evaluate \int_{-\infty}^{+\infty} \frac{cosmx\,dx}{(x^2+1)}\ where\ m > 0.$



Solution:

Consider

$$\int_{-\infty}^{+\infty} \frac{cosmx\,dx}{(x^2+1)} = \oint_C \frac{e^{imz}dz}{(z^2+1)}$$

Where C is a contour as shown in figure.

The function $f(z) = \frac{e^{imz}}{(z^2+1)}$ has simple pole at $z = \pm i$

Out of these two poles only $z = +i$ lies within the contour C

Now, from Cauchy residue theorem

$$\oint_C \frac{e^{imz}}{(z^2+1)}dz = 2\pi i \times Res[f(z), i)\,]$$

$$\int_\Gamma \frac{e^{imz}}{(z^2+1)}dz + \int_{-R}^{+R} \frac{e^{imz}}{(x^2+1)}dx = 2\pi i \times \lim_{z\to i}\left\{(z-i).\frac{e^{imz}}{(z-i)(z+i)}\right\}$$

$$\int_\Gamma \frac{e^{imz}}{(z^2+1)}dz + \int_{-R}^{+R} \frac{cosmx}{(x^2+1)}dx + i\int_{-R}^{+R} \frac{sinmx}{(x^2+1)}dx = \frac{\pi}{e^m}$$

Here

$$\int_\Gamma \frac{e^{imz}}{(z^2+1)}dz + \int_{-R}^{+R} \frac{cosmx}{(x^2+1)}dx = \frac{\pi}{e^m}$$

$$\Rightarrow I_1 + I_2 = \frac{\pi}{e^m} \qquad \text{-----------------(1)}$$

Evaluation of definite integrals: $\int_{-\infty}^{+\infty} F(x)\left\{\begin{matrix}\cos mx \\ \sin mx\end{matrix}\right\}dx$

Now

$$I_1 = \int_\Gamma \frac{e^{imz}}{(z^2+1)}dz = \int_\Gamma \frac{e^{imR(cos\theta+isin\theta)}Rie^{i\theta}d\theta}{R^2e^{2i\theta}+1}$$

$$|I_1| \le \int_\Gamma \frac{e^{|imR(cos\theta+isin\theta)|}\left|Rie^{i\theta}\right|d\theta}{|R^2e^{2i\theta}+1|}$$

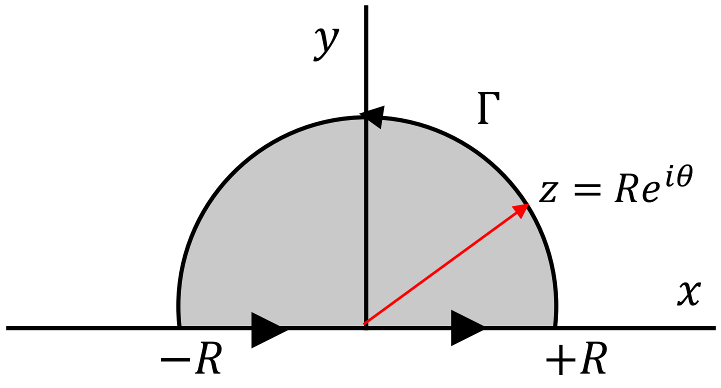
$$|I_1| \le \int_\Gamma \frac{e^{-Rsin\theta}Rd\theta}{|R^2e^{2i\theta}+1|}$$

Therefore when the limit $R\to\infty$, then I_1 becomes

$$\lim_{R\to\infty}|I_1| = \lim_{R\to\infty}\int_\Gamma \frac{e^{-Rsin\theta}Rd\theta}{|R^2e^{2i\theta}+1|} = 0$$

Now taking limit on both side of equation (1)

$$\lim_{R\to\infty}\int_\Gamma \frac{e^{imz}}{(z^2+1)}dz + \lim_{R\to\infty}\int_{-R}^{+R}\frac{e^{imz}}{(x^2+1)}dx = \frac{\pi}{e^m}$$



$$\Rightarrow \int_{-\infty}^{+\infty}\frac{e^{imz}}{(x^2+1)}dx = \frac{\pi}{e^m}$$

This is the required integration.

$$Also, \int_0^{+\infty}\frac{e^{imz}}{(x^2+1)}dx = \frac{\pi}{e^m}$$