3 (Sem–3/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3026

(Group Theory–I)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** questions : 1×10=10

   (a) What do you mean by the symmetry group of a plane figure ?

   (b) The set S of positive irrational numbers together with 1 is a group under multiplication. Justify whether it is true ***or*** false.

Contd.

(c) Define a binary operation on the set $\{0, 1, 2, 3, 4, 5\}$ for which it is a group.

(d) Let $G = \langle a \rangle$ be a cyclic group of order n. Write a necessary and sufficient condition for which a^k is a generator of G.

(e) What do you mean by even permutation? Give an example.

(f) Write the order of the alternating group of degree n.

(g) Let $G = S_3$ and $H = \{(1), (13)\}$. Write the left cosets of H in G.

(h) Show that there is no isomorphism from Q, the group of rational numbers under addition, to $Q^{\#}$, the group of non-zero rational numbers under multiplication.

(i) State Cayley's theorem.

(j) Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be defined by $\phi(x) = 3x$, $x \in \mathbb{Z}_{12}$. Find ker ϕ.

(k) On the set $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$, define a binary operation for which it is a group.

(l) Define normalizer of an element in a group G.

(m) Product of two subgroups of a group is again a subgroup. State whether true ***or*** false.

(n) State Lagrange's theorem.

(o) What is meant by external direct product of a finite number of groups?

(p) Find the order of the permutation

$$f = \begin{pmatrix} a & b & c & d & e \\ c & a & b & e & d \end{pmatrix}$$

(q) The subgroup of an abelian group is abelian. State whether it is true ***or*** false.

(r) Give the statement of third isomorphism theorem.

2. Answer ***any five*** questions : 2×5=10

(a) Show that in a group G, right and left cancellation laws hold.

(b) Show that a group of prime order is cyclic.

(c) Every subgroup of an abelian group is normal. Justify whether it is true ***or*** false.

(d) Let $\mathbb{C}^*$ denote the group of non-zero complex numbers under multiplication. Define $\phi : \mathbb{C}^* \to \mathbb{C}^*$ by $\phi(x) = x^4, x \in \mathbb{C}^*$. Show that ϕ is a homomorphism and find $\ker \phi$.

(e) If ϕ is an isomorphism from a group G onto a group $\overline{G}$, then show that ϕ carries the identity element of G to the identity element of $\overline{G}$.

(f) What is meant by cycle of a permutation ? Give an example.

(g) Show that in a group $(G, \bullet)$, $(a.b)^{-1} = b^{-1}.a^{-1}$, $a, b \in G$.

(h) Define centre of a group G and give an example.

(i) Give an example of a group containing only three elements.

(j) Define group isomorphism and give an example.

3. Answer ***any four*** questions : 5×4=20

(a) Show that *any two* cycles of a permutation of a finite set are disjoint.

(b) If H and K are two normal subgroups of a group G such that $H \cap K = \{e\}$ (e being the identity element of G), then show that $hk = kh$ for all $h \in H, k \in K$.

(c) Let H be a subgroup of a group G. Show that there exists a one-one and onto map between the set of all left cosets of H in G and the set of all right cosets of H in G.

(d) Let G be a group. If $a \in G$ is of finite order n and also $a^m = e$, then show that n/m.

(e) Let f be a homomorphism from a group G to a group G'. Show that ker f is a normal subgroup of G.

(f) If $\mathbb{R}^*$ is the group of non-zero real numbers under multiplication, then show that $(\mathbb{R}^*, \bullet)$ is not isomorphic to $(\mathbb{R}, +)$.

(g) Prove that a cyclic group is abelian.

(h) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a self mapping ϕ on G which is a homomorphism and justify your answer.

4. Answer **any four** questions : 10×4=40

(a) Let G be a group. Show that

(i) the centre of G is a subgroup of G;

(ii) for each $a \in G$, the centralizer of a is a subgroup of G.

(b) Let G be a group in which

$(ab)^3 = a^3b^3$

$(ab)^5 = a^5b^5$ for all $a, b \in G$.

Prove that G is abelian.

(c) Prove that every subgroup of a cyclic group is cyclic. Also show that if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n.

(d) If H and K are finite subgroups of a group G, then prove that

$$|HK| = \frac{|H| . |K|}{|H \cap K|}$$

(e) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

(f) Let G be a finite abelian group and let p be a prime that divides the order of G. Prove that G has an element of order p.

(g) Let ϕ be an isomorphism from a group G onto a group $\overline{G}$. Prove that—

(i) for every integer n and for every $a \in G$, $\phi(a^n) = [\phi(a)]^n$;

(ii) $|a| = |\phi(a)|$ for all $a \in G$.

(h) State and prove the second isomorphism theorem for groups.

(i) Show that the order of a cyclic group is same as the order of its generator.

(j) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Find all the subgroups of G and verify Lagrange's theorem for each subgroup.

———