2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : $1\times10=10$

(i) Write down the formulae of transformation from one pair of rectangular axes to another with same origin.

(ii) Find the equation to the locus of the point $P(t, 2t)$ if t is a parameter.

(iii) For what value of a, the transformation $x' = -x + 2,\ y' = ax + 3$ is a translation ?

(iv) What is the locus represented by the equation $ax^2 - 5xy + 6y^2 = 0$?

(v) Write down the polar equation of the straight line $x = 0$.

(vi) Under what condition

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

may represents a pair of straight lines ?

(vii) What will be the equation of the line $ax + by + c = 0$ if the origin is transferred to the point (α, β) ?

(viii) The parabola represented by the equation $y^2 = 4ax$ is not a closed curve. How can you justify it from the given equation ?

(ix) Write the relationship between the lengths of semi-major axis, semi-minor axis and the eccentricity for the standard equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b > a.$$

(x) What conic does the following equation represent ?

$$x^2 + 2xy + y^2 - 2x - 1 = 0$$

(xi) What are the direction ratios of the normal to the plane given by equation $ax + by + cz + d = 0$?

(xii) Write down the direction cosines of z-axis.

(xiii) When does the equation

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represent a sphere ?

(xiv) When is a plane said to be parallel to a line ?

(xv) Mention the condition under which the lines $\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$ and

$$\frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$$ are coplanar.

(xvi) What are centre and radius of the sphere given by the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0\ ?$$

(xvii) Define the polar plane of a point (α, β, γ) with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$.

(xviii) What are the coordinates of the vertex of the cone

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0 \ ?$$

2. Answer ***any five*** : 2×5=10

(a) Find the equation of the line $y = \sqrt{3}x$ when the axes are rotated through an angle $\frac{\pi}{3}$.

(b) If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the extremities of any focal chord of the parabola $y^2 = 4ax$, prove that $t_1 t_2 = -1$.

(c) If the two pair of lines

$x^2 - 2pxy - y^2 = 0$ and

$x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq + 1 = 0$.

(d) If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

(e) Find the equation of the plane containing the lines

$2x + 3y + 5z - 7 = 0$, $3x - 4y + z + 14 = 0$

and passing through the origin.

(f) Find the equation of the cone whose vertex is at the origin and whose guiding curve is given by

$x = a$, $y^2 + z^2 = b^2$.

(g) Find the equation of the sphere through the circle

$x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$

and the point $(1, 2, 3)$.

(h) Mention the conditions under which the general equation of the second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents *(i)* a parabola, *(ii)* an ellipse, *(iii)* a hyperbola, and *(iv)* a circle.

(i) Find the perpendicular distance of the point $(1, 4, -2)$ from the plane $2x - 3y + z = 5$.

(j) The axis of a right circular cylinder is $\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{2}$ and its radius is 5. Find its equation.

3. Answer **any four** : 5×4=20

(a) Prove that the transformation of rectangular axes which converts $\frac{X^2}{P}+\frac{Y^2}{Q}$ into $ax^2+2hxy+by^2$ will convert $\frac{X^2}{P-\lambda}+\frac{Y^2}{Q-\lambda}$ into

$$\frac{ax^2+2hxy+by^2-\lambda\left(ab-h^2\right)\left(x^2+y^2\right)}{1-(a+b)\lambda+\left(ab-h^2\right)\lambda^2}.$$

(b) Prove that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of parallel straight lines if $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$.

(c) Show that the line $lx+my=n$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $a^2l^2+b^2m^2=n^2$.

(d) Prove that the product of the perpendiculars from any point on a hyperbola to the asymptotes is constant.

(e) A plane passes through a fixed point (p, q, r), and cut the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$.

(f) Find the centre and radius of the circle

$$x^2+y^2+z^2-8x+4y+8z-45=0,$$
$$x-2y+2z=3.$$

(g) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by the lines drawn from O is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

(h) Find the equation of the circular cylinder whose guiding circle is $x^2+y^2+z^2=9,\ x-y+z=3$.

4. Answer ***any four*** : 10×4=40

(a) If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points, where these meet the axes is

$$ax^2-2hxy+by^2+2gx+2fy+c+\frac{4fg}{c}xy=0.$$

(b) Obtain the equation of the chord of the conic $\frac{l}{r}=1+e\cos\theta$, joining the two points on the conic, whose vectorial angles are $(\alpha+\beta)$ and $(\alpha-\beta)$.

(c) If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'}+\frac{1}{QS.SQ'}=\text{a constant.}$$

(d) Find the equation of a polar of a given point $P(x_1, y_1)$ with respect to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$.
Hence show that if the polar of a point P with respect to the conic passes through a point Q, then the polar of Q also passes through P.

(e) *(i)* Show that the locus of the points of intersection of perpendicular tangents to a parabola is its directrix.

(ii) Find the asymptotes of the hyperbola $xy+ax+by=0$.

(f) What do you mean by skew lines ? How do you define the shortest distance between two such lines ? Find the length and the equations of the line of shortest distance between the lines

$$3x-9y+5z=0,\ x+y-z=0$$
$$6x+8y+3z-13=0,\ x+2y+z-3=0.$$

(g) Prove that the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$$ cuts the

ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

$$b\sqrt{1-\frac{\lambda^2}{a^2-c^2}}.$$

(h) Prove that the lines through (α, β, γ) at right angles to their polars with respect to $\frac{x^2}{a+b}+\frac{y^2}{2a}+\frac{z^2}{2b}=1$ generate the cone

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

What is the peculiarity of the case when $a=b$?

(i) Show that the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and guiding curve is $x^2+2y^2=1, z=3$ is

$$3(x^2+2y^2+z^2)+8yz-2zx+6x-24y-18z+24=0.$$

(j) What do you mean by a director sphere? Find the equation of the director sphere of the conicoid $ax^2+by^2+cz^2=1$. Hence or otherwise prove that the director sphere of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

$$x^2+y^2+z^2=a^2+b^2+c^2.$$